PDE-methods for Asian options

Georgios Foufas

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Abstract

The main objectives of this thesis are to examine different PDE-methods for the pricing of Asian options and to make an adaptive finite element implementation estimating the value of the Asian option with both fixed and floating strike.

Two of the most important PDE methods are presented and explained. It is shown that the Asian option is a special case of an option on a traded account. The resulting PDE:s for the Asian option are of parabolic type with one space-dimension and can be applied to both continuous and discrete Asian options. The suggested adaptive finite element method is very stable and gives fast and accurate results.

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Chapter 1

Introduction

A contingent claim, or a derivate, is a contract the value of which depends on the values of other assets. One of the most common derivates is the European call option. A European call option on a given stock with strike price K and maturity date T is the right, but not the obligation, for the holder of the option to buy one share of the stock at the price K at the time T. A European put option with strike price K and time of maturity T gives the holder the right, but not the obligation, to sell one share of the stock at the price K at maturity. The so called American option differs from the European option so that the holder can exercise the option at any time prior to the maturity date. Calls and puts are often called vanilla options.

Stocks and options have a long history. Stocks have existed for at least 750 years. Option contracts were used already during the Middle Ages. Valuing financial derivates in a theoretical convincing way has been difficult throughout history. A very important contribution was given in 1973 when Black and Scholes presented their solution to the valuation of the European call option, based on the assumption that the stock log-price is governed by a so called Brownian motion. Their solution was based of the Itô calculus on Brownian motion. The concept arbitrage, that is risk free profit, is very central here. The most difficult part in this area is to understand the price dynamics of the underlying contracts.

Another kind of option is the exotic option with a payoff which does not just depend on its value on the maturity date, but on the history of the underlying asset price. There are many different kinds of exotic options. Some of them are easy to price and analytical pricing formulas exist, but most of them are more difficult to value. The average option, or the so called Asian option is an example of an option without a (known) closed form price formula. This paper focuses on PDE-methods for the pricing of Asian options.

Chapter 2

Asian options

The Asian option was invented by Phelim P. Boyle and David Emanuel in 1979, but The Journal of Finance rejected their paper since the asset was not traded at that time (private communication). Asian options are securities with payoffs which depend on the average of the underlying stock price over some time interval. They are commonly traded and are often relatively inexpensive compared to European calls. Asian options were introduced partly to avoid a problem common for European options, where the speculators could drive up the gains from the option by manipulating the price of the underlying asset near to the maturity date (see Bergman [2] or Wall Street Journal, Jan. 21, 1982, p. 4). The name Asian option probably originates from the Tokyo office of Bankers Trust, where it first was offered (see Nelken [13]).

Different kinds of averages are used, resulting in different types of Asian options, with different values. The method of sampling is also important. A continuous sampling may give easier calculations, but in reality the prices are mostly discretely sampled, and therefore discrete sampling is the most interesting case. The geometric Asian option with time of maturity T and strike price K has the payoff

$$\max\left(\prod_{k=1}^{N} S(t_k)^{1/N} - K, 0\right), \tag{2.1}$$

where $0 < t_1 < t_2 < \cdots < t_N = T$. For this option one can use the Black-Scholes framework to determine a closed-form pricing formula. Note that if $N = 1$ the option is reduced to a European call.
The average rate call with strike price K and time of maturity T has the

payoff

$$\max\left(\frac{1}{T}\int_0^T S(t)dt - K, 0\right), \tag{2.2}$$

while the discrete average rate call with strike price K and time of maturity T has the payoff

$$\max\left(\frac{1}{N}\sum_{k=1}^{N} S(t_k) - K, 0\right), \tag{2.3}$$

where $0 < t_1 < t_2 < \cdots < t_N = T$. There are no known closed-form pricing formulas for average rate options, but a variety of numerical techniques have been developed to find the corresponding prices.

The average rate call is cheaper than the European call at the writing date, see Table 2.1 and Theorem 6 in Chapter 3.

There are also variants of the Asian options mentioned above. For a floating strike Asian option the strike K in (2.2) and (2.3) is replaced by the spot price $S(T)$ at maturity. The corresponding options are often called average strike put and discrete average strike put respectively.

	Average rate call			European call		
$K\backslash\sigma$	0.10	0.20	0.30	0.10	0.20	0.30
90	13.73	14.14	15.24	14.63	16.70	19.70
100	5.26	7.04	9.06	6.81	10.45	14.23
110	0.73	2.70	4.86	2.17	6.04	10.02

Table 2.1: *The European call compared to the average rate call for various strikes K and volatilities σ when r=0.05, T=1 and t=0.*

Chapter 3

Underlying Theory

Throughout this section we are working in the time interval $0 \leq t \leq T$. Let B(t) denote the price of a risk free asset at time t governed by the equation $B(t) = B(0)e^{rt}$, where r is the constant interest rate. A common hypothesis about the behaviour of asset prices is that they are given by geometric Brownian motions which implies that the asset prices are log-normally distributed (see e.g. D. Duffie [6] or T. Björk [3]). The price $S(t)$ of an asset at time t, solves the following stochastic differential equation

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dW(t)), \\ S(0) &= S_0, \end{aligned} \tag{3.1}$$

where σ is the volatility, $\mu \in \mathbf{R}$ and $W(t)$ is a normalised Wiener process. Here σ is assumed to be a positive real number. The solution of (3.1) is

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}. \tag{3.2}$$

Now set

$$\tilde{W}(t) = \frac{\mu - r}{\sigma} t + W(t), \tag{3.3}$$

and note that

$$dS(t) = S(t)(r dt + \sigma d\tilde{W}(t)). \tag{3.4}$$

According to Cameron-Martin's theorem there exists another probability measure than the objective measure P, the risk neutral measure Q, such that $\tilde{W}$ is a Q-Wiener process. The solution of (3.4) equals

$$S(t) = S(0)e^{(r - \frac{\sigma^2}{2})t + \sigma \tilde{W}(t)}, \tag{3.5}$$

and the measures P and Q are equivalent. The existence of the risk neutral measure Q assures that the market is free of arbitrage possibilities.

Because the Wiener process is not differentiable in the usual sense, the equation (3.1) is interpreted in the sense of stochastic differential calculus initiated by K. Itô. The most fundamental tool in stochastic calculus, Itô's lemma is given below. But first we state a definition. If the stochastic process $(h(t))_{0\leq t\leq T}$ is progressively measurable and

$$\int_0^T \mid h(t) \mid^p dt < \infty \text{ almost surely,} \tag{3.6}$$

for some $p \in [1, \infty[$, then we say that h belongs to the class $L_W^p[0,T]$.

Itô's lemma. *Let the function $u(t, x_1, \ldots, x_m)$ be two times continuously differentiable in $x_1, \ldots, x_m \in \mathbf{R}$ and one time continuously differentiable in $t \in [0,T]$. Suppose we have m stochastic differentials*

$$dX_i(t) = a_i(t)dt + \sum_{k=1}^{n} b_{ik}(t)dW_k(t), \tag{3.7}$$

dependent on n stochastic independent Wiener Processes $W_1, \ldots, W_n$. Let $\mathcal{F}_t = \sigma(W_1(\lambda), \ldots, W_n(\lambda), \lambda \leq t)$. Let also the coefficients $a_i(t), b_{ik}(t)$ fulfil $a_i(t) \in L_W^1[0,T]$, $b_{ik}(t) \in L_W^2[0,T]$ and so, especially, for fixed t the processes are $\mathcal{F}_t$-measurable. Let also $X(t) = (X_1(t), \ldots, X_m(t))$. Then we have

$$\begin{aligned} du(t, X(t)) = {} & \frac{\partial u}{\partial t}(t, X(t))dt + \sum_{i=1}^{m} \frac{\partial u}{\partial x_i}(t, X(t))dX_i(t) \\ & + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial x_i \partial x_j}(t, X(t))dX_i(t)dX_j(t). \end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned} & dtdt = 0, \quad dtdW_i(t) = 0, \\ & dW_i(t)dW_i(t) = dt, \quad dW_i(t)dW_j(t) = 0 \quad \text{if } i \neq j. \end{aligned}$$

3.1 Derivation of the Black-Scholes formula

Let $v(t, S(t))$ denote the value of the portfolio at time t, with the terminal condition $v(T, S(T)) = g(S(T))$, where the function g is piecewise continuous and fulfils

$$\sup_{x \in \mathbf{R}} (e^{-C|x|} |g(e^x)|) < \infty \tag{3.9}$$

for an appropriate constant $C > 0$. We then say that $g \in \mathcal{P}$. Suppose that the process $(v(t, S(t))_{0 \le t \le T}$ is the value process of a self-financing strategy $\big(h_S(t), h_B(t)\big)_{0 \le t \le T}$ in the stock and the risk free asset, that is

$$v(t, S(t)) = h_S(t)S(t) + h_B(t)B(t), \tag{3.10}$$
$$dv(t, S(t)) = h_S(t)dS(t) + h_B(t)dB(t). \tag{3.11}$$

By applying Ito's lemma and using (3.11) we get,

$$\begin{aligned} dv(t, S(t)) &= v'_t(t, S(t))dt + v'_s(t, S(t))dS(t) + \frac{1}{2} v''_{ss}(t, S(t))(dS(t))^2 \\ &= h_S(t)dS(t) + rh_B(t)B(t)dt. \end{aligned} \tag{3.12}$$

Identifying coefficients in (3.12) yields $h_S = v'_s$. Rearranging the terms and using (3.10) we get the famous Black-Scholes differential equation

$$\begin{aligned} v'_t(t, S(t)) + \frac{\sigma^2 S(t)^2}{2} v''_{ss}(t, S(t)) + rS(t)v'_s(t, S(t)) - rv(t, S(t)) = 0, \\ t < T,\ S(t) > 0. \end{aligned} \tag{3.13}$$

Together with the terminal condition $v(T, S(T)) = g(S(T))$, equation (3.13) has the following solution,

$$v(t, S(t)) = e^{-r\tau} E\left[g(se^{(r - \frac{\sigma^2}{2})\tau + \sigma W(\tau)} \right], \tag{3.14}$$

where $s = S(t)$ and $\tau = T - t$. Observe that (3.14) is independent of the drift coefficient μ.

We thus have the following important result.

Theorem 1. *Let $g \in \mathcal{P}$. A simple European derivate with payoff $Y = g(S(T))$ at maturity T has the theoretical value $v(t, S(t))$ at time t, where*

$$v(t, S(t)) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma W(\tau)})\right], \tag{3.15}$$

and $\tau = T - t$.

We can simplify (3.15) using the risk neutral measure Q (see Geman, Karoui and Rochet [10], for a detailed discussion about changes of probability measure).

Theorem 2. *The value $v(t, S(t))$ is equal to*

$$e^{-r\tau} E^Q[g(S(T)) \mid \mathcal{F}_t].$$

Proof. According to (3.5) we have $S(T) = S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(\tilde{W}(T)-\tilde{W}(t))}$ and hence

$$E^Q[g(S(T)) \mid \mathcal{F}_t] = E^Q\left[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(\tilde{W}(T)-\tilde{W}(t))}) \mid \mathcal{F}_t\right]. \tag{3.16}$$

But since $(\tilde{W}(T) - \tilde{W}(t))$ and $\mathcal{F}_t$ are stochastic independent and $\tilde{W}$ is a Q-Brownian motion, the right hand side of (3.16) becomes

$$E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma(W(T)-W(t))})\right]_{|s=S(t)} = e^{r\tau} v(t, S(t)),$$

which proves the theorem.

We now state the famous Black-Scholes formula which gives the value of a European call option with payoff $Y = \max(0, S(T) - K)$ at maturity T.

Theorem 3 (Black-Scholes formula). *A European call option with maturity date T and strike price K has the value $c(t, S(t), K)$ at time $t < T$ where*

$$c(t, s, K) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{3.17}$$
$$d_1 = \frac{\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

and where Φ is the probability distribution function for a $N(0, 1)$ distributed stochastic variable.

Proof. Theorem 1 gives that

$$c(t,s,K) = e^{-r\tau} E\left[\max\left(0, se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}G} - K\right)\right],$$

where $G \in N(0,1)$. From this it follows that

$$\begin{aligned} c(t,s,K) &= e^{-r\tau} E\left[se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}G} - K; \quad G \le \frac{\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right] \\ &= e^{-r\tau}\left(E\left[se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}G}; \quad G \le d_2\right] - K\Phi(d_2)\right). \end{aligned}$$

Here

$$\begin{aligned} e^{-r\tau} E\left[se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}G}; \quad G \le d_2\right] &= s\int_{x \le d_2} e^{\frac{-\sigma^2}{2}\tau - \sigma\sqrt{\tau}x - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ = s\int_{x\le d_2} e^{\frac{-(\sigma\sqrt{\tau}+x)^2}{2}} \frac{dx}{\sqrt{2\pi}} &= s\Phi(\sigma\sqrt{\tau} + d_2) = s\Phi(d_1), \end{aligned}$$

which proves the theorem.

The price of the European put option can be derived in the same manner as the call price. Alternatively to derive the European put price one can use the so called call-put parity relation.

Theorem 4 (Call-put parity). *Let c and p be the value of an European call and put option respectively. Then we have*

$$p(t,s,K,T) = Ke^{-r\tau} - s + c(t,s,K,T). \tag{3.18}$$

Using Theorems 3 and 4 we can easily calculate the price of an European put option, p(t, s, K, T).

$$\begin{aligned} p(t,s,K,T) &= Ke^{-r\tau} - s + s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \\ &= Ke^{-r\tau}\Phi(-d_2) - s\Phi(-d_1). \end{aligned} \tag{3.19}$$

3.2 General derivate valuation formula

To be able to handle more complex derivates we extend the previous valuation formula in Theorem 2 to European derivates with payoff $X \in L^2(Q)$ and state the following theorem (for a more detailed discussion see Borell [4]).

Theorem 5. *A European derivate with payoff $X \in L^2(Q)$ at maturity T has the theoretical value*

$$v(t) = e^{-r\tau} E^Q[X \mid \mathcal{F}_t]. \tag{3.20}$$

Theorem 5 has the following interesting consequence (cf. the Geman and Yor paper [11]).

Theorem 6. *If $\rho(t) \geq 0$ and $\int_0^T \rho(t)dt = 1$, then for any $T > 0$,*

$$e^{-rT} E^Q \left[\left(\int_0^T S(\lambda)\rho(\lambda)d\lambda - K \right)^+ \mid \mathcal{F}_0 \right] < e^{-rT} E^Q \left[\left(S(T) - K \right)^+ \mid \mathcal{F}_0 \right].$$

Proof. Note that $E^Q[X \mid \mathcal{F}_0] = E^Q[X]$ so we omit the σ-algebra $\mathcal{F}_0$ in the following. Note also that

$$E^Q \left[\left(S(T_0) - K \right)^+ \right] < E^Q \left[\left(S(T) - K \right)^+ \right], \text{ if } T_0 < T$$

since an American call price is the same as the price of the corresponding European call when the underlying stock does not pay dividends. Now

$$\begin{aligned}
&E^Q \left[\left(\int_0^T S(\lambda)\rho(\lambda)d\lambda - K \right)^+ \right] = E^Q \left[\left(\int_0^T \left(S(\lambda) - K \right)\rho(\lambda)d\lambda \right)^+ \right] \\
&\leq E^Q \left[\int_0^T \left(S(\lambda) - K \right)^+ \rho(\lambda)d\lambda \right] = \int_0^T \rho(\lambda) E^Q \left[\left(S(\lambda) - K \right)^+ \right] d\lambda \\
&< \int_0^T \rho(\lambda) E^Q \left[\left(S(T) - K \right)^+ \right] d\lambda = \int_0^T \rho(\lambda)d\lambda E^Q \left[\left(S(T) - K \right)^+ \right] \\
&= E^Q \left[\left(S(\lambda) - K \right)^+ \right]
\end{aligned}$$

and Theorem 6 follows at once.

Chapter 4

The Ingersoll and Rogers-Shi approaches to Asian options

No general analytical price formula is known for the average rate option, on the other hand several approximations that produce closed form expressions have appeared. Thus Geman and Yor computed the Laplace transform of the Asian option price, but numerical inversion remains problematic for low volatility and short maturity cases (see Fu Madan and Wang [9]). Monte Carlo simulation works well, but sometimes it is computational expensive.

4.1 Two PDE methods for the pricing of Asian options

In general, the price of an Asian option can be found by solving a PDE in two space dimension as noted by Ingersoll [12]. Assuming a payoff

$$\max\left(S(T) - \frac{1}{T}\int_0^T S(t)dt, 0\right) \tag{4.1}$$

Ingersoll introduces the space variables $A(t) = \int_0^t S(\lambda)d\lambda$ and S(t), and postulates that the price at time t is given by $v(t, S(t), A(t))$. Since the differential $dA(t) = S(t)dt$ is deterministic, a risk less hedge for the option only requires an elimination of the stock-induced risk, which is accomplished by holding v'_s shares of stock for each option. Ingersoll therefore suggests the following pricing equation for the Asian option:

$$
\begin{aligned}
v'_t + rsv'_s + sv'_a + \frac{1}{2}\sigma^2 s^2 v''_{ss} - rv &= 0, \\
v(t,0,a) &= 0, \\
v'_s(t,\infty,a) &= 1, \\
v(t,s,\infty) &= 0, \\
v(T,s,a,) &= \max\left(0, s - \frac{a}{T}\right).
\end{aligned} \tag{4.2}
$$

The differential equation (4.2) is valid for all $a > 0$ and $s > 0$. In fact, given $a > 0$ and $s > 0$, $P\big[|A(t) - a| < \epsilon, |S(t) - s| < \epsilon\big] > 0$, for all $0 < t \le T$ and $\epsilon > 0$. Ingersoll also notes that the change of variable v(s,a,t)=aG(t,x), where x=s/a, gives a one-dimensional PDE for the floating strike Asian option,

$$
\begin{aligned}
\frac{1}{2}\sigma^2 x^2 G''_{xx} + (rx - x^2)G'_x + (x - r)G + G'_t &= 0, \\
G(t,0) &= 0, \\
G'_x(t,\infty) &= 1, \\
G(T,x) &= \max\left(0, x - \frac{1}{T}\right).
\end{aligned} \tag{4.3}
$$

A more rigorous proof to (4.2) is now presented (see e.g. Björk [3] or Borell [4]). Consider a self financing strategy in the stock and risk less asset as before

$$
v(t,S(t),A(t)) = h_S(t)S(t) + h_B(t)B(t), \tag{4.4}
$$

$$
dv(t,S(t),A(t)) = h_S(t)dS(t) + h_B(t)dB(t). \tag{4.5}
$$

Applying Ito's lemma we get,

$$
\begin{aligned}
dv(t,S(t),A(T)) = {} & v'_t(t,S(t),A(t))dt + v'_s(t,S(t),A(t))dS(t) \\
& + v'_a(t,S(t),A(t))dA(t) + \frac{1}{2}v''_{ss}(t,S(t),A(t))(dS(t))^2.
\end{aligned} \tag{4.6}
$$

In view of (4.5) this implies that

$$
h_S(t) = v'_s(t,S(t),A(t)),
$$

from which it follows that

$$
\begin{aligned}
& v'_t(t,S(t),A(t))dt + v'_a(t,S(t),A(t))dA(t) + \frac{1}{2}v''_{ss}(t,S(t),A(t))(dS(t))^2 \\
& \quad = h_B(t)dB(t) = rh_B(t)B(t)dt \\
& \quad = r\big(v(t,S(t),A(t)) - v'_s(t,S(t),A(t))S(t)\big)dt.
\end{aligned} \tag{4.7}
$$

Equation (4.7) is satisfied if

$$v'_t + rsv'_s + sv'_a + \frac{1}{2}\sigma^2 s^2 v''_{ss} - rv = 0, \tag{4.8}$$

which is exactly the same as (4.2).

Rogers and Shi [14] presented a one-dimensional PDE that can model both fixed and floating strike Asian options. They also computed lower and upper bounds for the price of the Asian option, where the lower bound is very accurate. They worked with a stock evolving according to (3.4) (for simplicity we drop the "~" notation, and let W be a Brownian motion relative to the martingale measure Q) and used a model were the option has the payoff

$$X = \left(\int_0^T S_u \mu(du) - K \right)^+, \tag{4.9}$$

where μ is a finite measure in $[0, T]$. Different options are achieved by the choice of the measure μ. Let δ_T denote the delta function at time T, that is

$$\int_0^T f(u) d\delta_T(u) = f(T), \quad f \in C([0, T]). \tag{4.10}$$

If $\mu(du) = T^{-1} I_{[0,T]}(u)du$, where $I_{[0,T]}$ is the indicator function of the interval $[0, T]$, the average rate call is achieved. If we take $\mu(du) = T^{-1} I_{[0,T]}(u)du - \delta_T(du)$ together with $K = 0$ we get an average strike put and if we take $\mu(du) = \delta_T(du)$, then we have an ordinary European call option. If we let $0 < t_1 < \cdots < t_n = T$ and $\mu(du) = \frac{1}{n} \sum_{k=1}^n \delta_{t_k}(du)$ we have a discrete average rate call. According to Theorem 5 in Section 3.2, the price of the option at time t is given by

$$e^{-r\tau} E^Q[X \mid \mathcal{F}_t],$$

where $\mathcal{F}_t = \sigma(W_\lambda,\ \lambda \leq t)$.

First assume $\mu(du) = \rho_u du$, where ρ is piecewise continuous. At the very end of this section we will comment on more general cases. Let now $M_t = E^Q[X \mid \mathcal{F}_t]$ and let $X = E[X] + \int_0^T \Psi(t) dW(t)$, where $X \in L^2_W[0, T]$. Then

$$M_t = E^Q[X \mid \mathcal{F}_t] = E[X] + \int_0^t \Psi(\lambda) dW(\lambda),\ 0 \leq t \leq T,$$

is a martingale and

$$dM_t = \Psi(t) dW(t). \tag{4.11}$$

Equation (4.9) now gives

$$
\begin{aligned}
M_t &= E^Q\left[\left(\int_0^T S_u\mu(du) - K\right)^+ \mid \mathcal{F}_t\right] \qquad (4.12)\\
&= S_t E^Q\left[\left(\int_t^T \frac{S_u}{S_t}\mu(du) - \frac{K-\int_0^t S_u\mu(du)}{S_t}\right)^+ \mid \mathcal{F}_t\right]\\
&= S_t E^Q\left[\left(\int_t^T \frac{S_u}{S_t}\mu(du) - x\right)^+\right]_{|x=\frac{K-\int_0^t S_u\mu(du)}{S_t}}\\
&= S_t\phi\left(t, \frac{K-\int_0^t S_u\mu(du)}{S_t}\right),
\end{aligned}
$$

where

$$
\phi(t,x) = E^Q\left[\left(\int_t^T \frac{S_u}{S_t}\mu(du) - x\right)^+\right]. \qquad (4.13)
$$

Let

$$
\xi_t = \frac{K-\int_0^t S_u\mu(du)}{S_t}, \qquad (4.14)
$$

and thus $M_t = S_t\phi(t,\xi_t)$. We now want to calculate the stochastic differential dM_t and in order to do so we need $\mathrm{d}\xi_t$ and therefore also $d(\frac{1}{S_t})$. By Itô's lemma

$$
\begin{aligned}
d\left(\frac{1}{S_t}\right) &= -\frac{1}{S_t^2}dS_t + \frac{1}{S_t^3}(dS_t)^2 = -\frac{1}{S_t}\big(rdt + \sigma dW_t\big) + \frac{1}{S_t^3}\sigma^2 S_t^2 dt \qquad (4.15)\\
&= \frac{1}{S_t}\Big((\sigma^2 - r)dt - \sigma dW_t\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
d\xi_t &= d\Big(\frac{1}{S_t}(K - \int_0^t S_u\mu(du))\Big) \qquad (4.16)\\
&= \frac{1}{S_t}\Big((\sigma^2 - r)dt - \sigma dW_t\Big)\Big(K - \int_0^t S_u\mu(du)\Big) + \frac{1}{S_t}(-S_t\rho_t dt)\\
&= \xi_t\Big((\sigma^2 - r)dt - \sigma dW_t\Big) - \rho_t dt
\end{aligned}
$$

Again using Itô's lemma we get

$$\begin{aligned}
dM_t &= d(S_t\phi(t,\xi_t)) = S_t d\phi(t,\xi_t) + \phi(t,\xi_t)dS_t + dS_t d\phi(t,\xi_t) \qquad (4.17)\\
&= S_t\Big(\phi'_t(t,\xi_t)dt + \phi'_x(t,\xi_t)d\xi_t + \frac{1}{2}\phi''_{xx}(t,\xi_t)(d\xi_t)^2\Big)\\
&\quad + \phi(t,\xi_t)S_t(rdt + \sigma dW_t) + dS_t d\phi(t,\xi_t)\\
&= S_t\Big(r\phi(t,\xi_t) + \phi'_t(t,\xi_t) - \phi'_x(t,\xi_t)(\rho_t + r\xi_t) + \frac{\sigma^2\xi_t^2}{2}\phi''_{xx}(t,\xi_t)\Big)dt\\
&\quad + \sigma S_t\big(\phi(t,\xi_t) - \xi_t\phi'_x(t,\xi_t)\big)dW_t\\
&= 0dt + \Psi(t)dW(t),
\end{aligned}$$

where the last equality follows from (4.11). Identifying we get

$$\begin{cases} r\phi(t,x) + \phi'_t(t,x) - \phi'_x(t,x)(\rho_t + rx) + \frac{\sigma^2x^2}{2}\phi''_{xx}(t,x) = 0,\\ \forall x \in \mathbf{R},\ \forall t > 0. \end{cases}$$

The transformation $f(t,x) = e^{-r(T-t)}\phi(t,x)$ gives

$$f'_t - (\rho_t + rx)f'_x + \frac{\sigma^2x^2}{2}f''_{xx} = 0, \qquad (4.18)$$

which is the one-dimensional PDE Rogers and Shi derived for Asian options. The boundary conditions is

$$f(T,x) = x^-, \qquad (4.19)$$

if $\mu(du) = \rho_u du$ with ρ piecewise continuous. This PDE in (4.18) is difficult to solve numerically since the diffusion term is very small. Zvan, Forsyth and Vetzal [17] suggest a method based on computational fluid dynamics techniques to overcome this difficulty. In [1] Andreasen applied the Rogers-Shi reduction to the discrete Asian option with very good results.

If the measure μ in (4.9) contains point masses (4.18) has to be interpreted in an appropriate way. A paper by Večeř [16], which we will discuss below, motivates the following change of variables

$$\begin{cases} t = t\\ z = q_t - x\\ u(t,z) = f(t,x) \end{cases} \qquad (4.20)$$

in (4.18), where

$$q_t = \int_t^T \rho_u du = \mu([t,T]). \qquad (4.21)$$

Now

$$\frac{\partial f}{\partial t} = \frac{\partial u}{\partial t} - \rho_t \frac{\partial u}{\partial z}, \quad \frac{\partial f}{\partial x} = -\frac{\partial u}{\partial z}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 u}{\partial z^2} \tag{4.22}$$

and (4.18) reduces to

$$\frac{\partial u}{\partial t} + r(q_t - z)\frac{\partial u}{\partial z} + \frac{\sigma^2}{2}(q_t - z)^2 \frac{\partial^2 u}{\partial z^2} = 0. \tag{4.23}$$

If (4.19) holds then $u(T, z) = z^+$. Note that the equation (4.23) makes sense if μ is a finite measure on $[0, T]$ with $q_t = \mu([0, T])$, $0 \leq t \leq T$.

Let $0 < \Theta < T$ and let ν be a finite measure on $[0, T]$ without atoms in the interval $]\Theta, T]$. Suppose $a \in]\Theta, T]$ and set $\mu = \mu_a = \nu + \delta_a$. Then the option price is a continuous function of $a \in]\Theta, T]$. Hence the final condition $u(T, z) = z^+$ is correct even if $\mu(\{T\}) \neq 0$.

In a forthcoming master thesis E. Broman [5] among other things will give an alternative derivation of equation (4.23) based on (4.8).

Chapter 5

Options on a traded account

This article mainly focuses on a method developed by Večeř [16]. Večeř notes that the Asian option is a special case of the option on a traded account. An option on a traded account is a contract which allows the holder of the option to switch among various positions in the underlying stock. The holder accumulates gains and losses resulting from this trading, and at maturity he keeps any gain and is forgiven any loss. Suppose that the stock evolves under the risk neutral measure Q according to

$$dS(t) = S(t)(rdt + \sigma dW(t)). \tag{5.1}$$

Denote the option holder's trading strategy, the number of shares held at time t, by $q_t \in [\alpha_t, \beta_t]$, where $\alpha_t \leq \beta_t$. Večeř uses a model were X_t the value of the option holder's account at time t satisfies

$$dX_t = \nu(X_t - q_t S_t)dt + q_t dS_t, \quad 0 \leq t \leq T, \tag{5.2}$$

where X_0 the initial wealth is given. Equation (5.2) contains a deterministic and a random term. The first term describes the growth in the cash position $X_t - q_t S_t$, due to the the addition of interest at rate ν, and the second term is due to the change in value of the stock holding. If $\nu = r$ then the trading strategy is self financing. In the case of Asian options q_t is a given deterministic function and $\alpha_t = \beta_t$. At maturity T the holder of the option will receive the payoff $[X_T]^+$. The seller of the option must be prepared to hedge against all possible strategies of the holder of the option. Therefore the price of the contract at time t should be the maximum over all possible strategies q_t of the discounted expected value of the payoff of the option, i.e.,

$$V(t, S_t, X_t) = \max_{\substack{\alpha_t \leq q_t \leq \beta_t \\ 0 \leq t \leq T}} e^{-r(T-t)} E^Q[(X_T)^+ \mid \mathcal{F}_t], \quad 0 \leq t \leq T. \tag{5.3}$$

By choosing $\alpha_t = \beta_t = 1$ we obtain the European call in several ways. For example, if $\nu = r$ and $X_0 = S_0 - Ke^{-rT}$, then $X_T = S_T - K$. If $\nu = 0$ and $X_0 = S_0 - K$, we also have $X_T = S_T - K$. In a similar way, if $\alpha_t = \beta_t = -1$, we obtain the European put.

It is possible to view a variety of contingent claims as options on traded accounts, as for example the so called passport option and vacation option. Below we will stress on the Asian option. By slightly extending the definition of the option on a traded account it is also possible to incorporate certain American styled options, as for example the American put. For more information see Shreve and Večeř [15] and Večeř [16].

5.1 The Asian option

Asian options can be considered a special case of options on a traded account. When we study Asian options no interest is added or charged from the traded account, i.e., $\nu = 0$. The equation for the traded account (5.2) then reduces to

$$dX_t = q_t dS_t = q_t S_t (r dt + \sigma dW_t), \quad 0 \leq t \leq T. \tag{5.4}$$

Now suppose

$$q_t = \int_t^T \rho(\lambda) d\lambda = \mu([t, T]), \tag{5.5}$$

for an appropriate piecewise continuous function ρ and a set $\mu(dt) = \rho(t)dt$. We then have

$$dX_t = \left(\int_t^T \rho(\lambda) d\lambda \right) dS_t, \quad 0 \leq t \leq T, \tag{5.6}$$

which gives

$$\begin{aligned} X(T) &= X(0) + \int_0^T \left(\int_t^T \rho(\lambda) d\lambda \right) dS_t \\ &= X(0) + \left[\int_t^T \rho(\lambda) d\lambda S(t) \right]_0^T + \int_0^T S(t)\rho(t)dt \\ &= X(0) - \left(\int_0^T \rho(\lambda) d\lambda \right) S(0) + \int_0^T S(t)\rho(t)dt \\ &= X(0) - \mu([0, T])S(0) + \int_0^T S(t)\mu(dt). \end{aligned} \tag{5.7}$$

For the average rate call and the average strike call we let

$$X(0) = \left(\int_0^T \rho(\lambda)d\lambda\right) S(0) - K = \mu([0,T])S(0) - K, \tag{5.8}$$

and for the average rate put and the average strike put we let

$$X(0) = \left(\int_0^T \rho(\lambda)d\lambda\right) S(0) + K = \mu([0,T])S(0) + K. \tag{5.9}$$

Thus for the call options equation (5.7) reduces to

$$X(T) = \int_0^T S(t)\rho(t)dt - K = \int_0^T S(t)\mu(dt) - K, \tag{5.10}$$

and for the put options we have

$$X(T) = K + \int_0^T S(t)\rho(t)dt = K + \int_0^T S(t)\mu(dt). \tag{5.11}$$

Let $\overline{S}_T = \frac{1}{T}\int_0^T S_t dt$. By taking $\rho = \frac{1}{T}$ in (5.10) the average rate call is achieved. In this case we have $q_t = 1 - \frac{t}{T}$, thus the average of the stock price could be achieved be selling off one share of stock at the constant rate $\frac{1}{T}$ shares per unit time. If we let $\rho = -\frac{1}{T}$ in (5.11), we get the average rate put

$$X(T) = K - \overline{S}_T. \tag{5.12}$$

For the average strike call with payoff $(S_T - \overline{S}_T)^+$, by a limit argument we conclude that $\mu(dt) = \delta_T(dt) - \frac{dt}{T}$ and $K = 0$, and for the average strike put with payoff $(\overline{S}_T - S_T)^+$, we have $\mu(dt) = \frac{dt}{T} - \delta_T(dt)$ and $K = 0$.

The discrete average rate call option is achieved by taking

$$\mu = \frac{1}{N}\sum_{k=1}^{N} \delta_{t_k}, \tag{5.13}$$

where $0 < t_1 < t_2 < \cdots < t_N = T$. Setting $A(T) = \frac{1}{N}\sum_{k=1}^{N} S_{t_k}$, equation (5.10) gives

$$X_T = \frac{1}{N}\sum_{k=1}^{N} S_{t_k} - K = A(T) - K. \tag{5.14}$$

Similarly we get the discrete average rate put by choosing

$$\mu = -\frac{1}{N}\sum_{k=1}^{N} \delta_{t_k}. \tag{5.15}$$

The discrete average strike call is achieved in the same manner by choosing $K = 0$ in (5.10) and

$$\mu = \delta_T - \frac{1}{N} \sum_{k=1}^{N} \delta_{t_k}. \tag{5.16}$$

Asian option type	Payoff	$\mu(dt)$	X_0
average rate call	$(\overline{S}_T - K)^+$	$\frac{dt}{T}$	$S_0 - K$
average rate put	$(K - \overline{S}_T)^+$	$-\frac{dt}{T}$	$K - S_0$
average strike call	$(S_T - \overline{S}_T)^+$	$\delta_T(dt) - \frac{dt}{T}$	0
average strike put	$(\overline{S}_T - S_T)^+$	$\frac{dt}{T} - \delta_T(dt)$	0
discr. av. rate call	$(A(T) - K)^+$	$\frac{1}{N}\sum_{k=1}^{N} \delta_{t_k}(dt)$	$S_0 - K$
discr. av. rate put	$(K - A(T))^+$	$-\frac{1}{N}\sum_{k=1}^{N} \delta_{t_k}(dt)$	$K - S_0$
discr. av. strike call	$(S(T) - A(T))^+$	$\delta_T(dt) - \frac{1}{N}\sum_{k=1}^{N} \delta_{t_k}(dt)$	0
discr. av. strike put	$(A(T) - S(T))^+$	$\frac{1}{N}\sum_{k=1}^{N} \delta_{t_k}(dt) - \delta_T(dt)$	0

Table 5.1: *Asian options as options on a traded account.*

5.2 Derivation of a pricing PDE

We now want to show that the value of the Asian option, $V(t, s, x)$ satisfies the following PDE

$$-rv + V_t' + rsV_s' + qrsV_x' + \frac{\sigma^2 s^2}{2}\left(V_{ss}'' + 2qV_{sx}'' + q^2 V_{xx}''\right) = 0. \tag{5.17}$$

Now suppose (5.17) is true. From Ito's Lemma it follows that

$$\begin{aligned} d\left[e^{-r(t-t_0)}V(t, S_t, X_t)\right] &= \\ &= -re^{-r(t-t_0)}V(t, S_t, X_t)dt + e^{-r(t-t_0)}dV(t, S_t, X_t) \\ &= e^{-r(t-t_0)}\Big(- rV(t, S_t, X_t)dt + dV(t, S_t, X_t)\Big). \end{aligned} \tag{5.18}$$

Using Itô's Lemma and (5.4), the right hand side of (5.18) becomes

$$\begin{aligned}
&e^{-r(t-t_0)}\Big(- rV dt + V_t' dt + V_s' dS_t + V_x' dX_t \\
&\qquad + \frac{1}{2}\left[V_{ss}''(dS_t)^2 + 2V_{sx}'' dS_t dX_t + V_{xx}''(dX_t)^2\right]\Big) \\
=&e^{-r(t-t_0)}\Big(- rV + V_t' + rS_tV_s' + rq_tS_tV_x' \\
&\qquad + \frac{1}{2}\left[\sigma^2 S_t^2 V_{ss}'' + 2\sigma^2 q_t S_t^2 V_{sx}'' + \sigma^2 q_t^2 S_t^2 V_{xx}''\right]\Big) dt \\
&\qquad\quad + e^{-r(t-t_0)}\big(\sigma S_t V_s' + \sigma q_t S_t V_x'\big) dW(t),
\end{aligned}$$

which according to the assumption (5.17) equals

$$e^{-r(t-t_0)}\big(\sigma S_t V_s' + \sigma q_t S_t V_x'\big) dW(t).$$

Thus we have

$$d\left[e^{-r(t-t_0)}V(t, S_t, X_t)\right] = e^{-r(t-t_0)}\big(\sigma S_t V_s' + \sigma q_t S_t V_x'\big) dW(t). \tag{5.19}$$

Integrating (5.19) yields

$$\begin{aligned}
&e^{-r(T-t_0)}V(T, S_T, X_T) - V(t_0, S_{t_0}, X_{t_0}) \\
&= \int_{t_0}^{T} e^{-r(t-t_0)}\big(\sigma S_t V_s' + \sigma q_t S_t V_x'\big) dW(t),
\end{aligned} \tag{5.20}$$

where the integral on the right hand side equals zero if enough integrability is assumed. Taking expectations we get

$$V(t_0, S_{t_0}, X_{t_0}) = e^{-r(T-t_0)}E^Q\left[V(T, S_T, X_T) \mid \mathcal{F}_{t_0}\right], \tag{5.21}$$

where $\mathcal{F}_{t_0} = \sigma(W(\lambda),\ \lambda \leq t_0)$. We therefore have

$$V(t, S_t, X_t) = e^{-r(T-t)}E^Q\left[V(T, S_T, X_T) \mid \mathcal{F}_t\right], \tag{5.22}$$

which is the usual expression for the value of a derivate. We therefore conclude that (5.17) is the correct pricing PDE for Asian options.

If $V(T, s, x) = x^+$, it especially follows that

$$V(t, S_t, X_t) = e^{-r(T-t)}E^Q\left[X_T^+ \mid \mathcal{F}_t\right]. \tag{5.23}$$

We can use the change of variable

$$Z_t = \frac{X_t}{S_t}, \tag{5.24}$$

to reduce the dimensionality of (5.17). To see this, we first note that Itô's lemma gives that

$$dZ_t = d\left(X_t \frac{1}{S_t}\right) = \frac{1}{S_t} dX_t + X_t d(\frac{1}{S_t}) + dX_t d(\frac{1}{S_t}). \tag{5.25}$$

According to (4.15) we have

$$d\left(\frac{1}{S_t}\right) = \frac{1}{S_t}\Big((\sigma^2 - r)dt - \sigma dW_t\Big), \tag{5.26}$$

which gives

$$\begin{aligned} dZ_t &= q_t(rdt + \sigma dW_t) + Z_t\Big((\sigma^2 - r)dt - \sigma dW_t\Big) - \sigma^2 q_t dt \\ &= (r - \sigma^2)(q_t - Z_t)dt + \sigma(q_t - Z_t)dW_t. \end{aligned} \tag{5.27}$$

From here on we suppose that Q is the Wiener-measure in $C[0,T]$ and that $W(t) = W(t,\omega) = \omega(t)$, $\omega \in C[0,T]$. We next define the process

$$\overline{W}_t = -\sigma t + W_t, \tag{5.28}$$

so that

$$dZ_t = r(q_t - Z_t)dt + \sigma(q_t - Z_t)d\overline{W}_t. \tag{5.29}$$

Let now $h(t) = \sigma$, $0 \leq t \leq T$, and

$$a(t) = \sigma t = \int_0^t h(\lambda)d\lambda, \quad 0 \leq t \leq T. \tag{5.30}$$

Then according to Cameron-Martin's theorem

$$dQ_a(\omega) = e^{\int_0^T h(t)dW(t) - \frac{1}{2}\int_0^T h^2(t)dt} dQ(\omega), \tag{5.31}$$

where $Q_a(A) = Q(A - a)$, if A belongs to the Borel-σ-algebra $\mathcal{B}(C[0,T])$. Notice that

$$dQ_a(\omega) = e^{\sigma W_T - \frac{\sigma^2}{2}T} dQ(\omega), \tag{5.32}$$

and that

$$Q_a(\overline{W} \epsilon A) = Q_a(-a + W \epsilon A) = Q_a(A + a) = Q(A) = Q(W \epsilon A). \tag{5.33}$$

It follows that $\overline{W}$ is a Brownian motion relative to the measure Q_a. Let $\overline{Q} = Q_a$ and $D_t = e^{\sigma W_t - \frac{\sigma^2}{2}t}$, so that

$$d\overline{Q}(\omega) = D_T(\omega) dQ(\omega). \tag{5.34}$$

Now suppose that

$$\begin{cases} \frac{\partial u}{\partial t} + r(q_t - z)\frac{\partial u}{\partial z} + \frac{\sigma^2}{2}(q_t - z)^2 \frac{\partial^2 u}{\partial z^2} = 0, \\ u_T = z^+. \end{cases} \tag{5.35}$$

Then according to Itô's lemma, equation (5.29) and the assumption (5.35) we have

$$du(t, Z(t)) = \sigma(q_t - Z_t)\frac{\partial u}{\partial z}(t, Z(t)) d\overline{W}(t), \tag{5.36}$$

which after integration yields

$$u(T, Z(T)) = u(t, Z(t)) + \int_t^T \sigma(q_\lambda - Z_\lambda)\frac{\partial u}{\partial z}(\lambda, Z(\lambda)) d\overline{W}(\lambda). \tag{5.37}$$

Taking expectations conditional on $\mathcal{F}_t$ and assuming that the integral on the right belongs to $L^2(dt \times d\overline{Q})$ we get

$$u(t, Z_t) = E^{\overline{Q}}\left[Z_T^+ \mid \mathcal{F}_t\right]. \tag{5.38}$$

We now want to show that

$$V(t, S_t, X_t) = S_t u\big(t, \frac{X_t}{S_t}\big), \tag{5.39}$$

or equivalently

$$e^{-r(T-t)} E^Q\left[X_T^+ \mid \mathcal{F}_t\right] = S_t E^{\overline{Q}}\left[Z_T^+ \mid \mathcal{F}_t\right]. \tag{5.40}$$

Suppose that $f \geq 0$ is $\mathcal{F}_t$ measurable, where

$$\mathcal{F}_t = \sigma(\overline{W}(\lambda),\ \lambda \leq t) = \sigma(W(\lambda),\ \lambda \leq t).$$

Then it is sufficient to show that

$$e^{-r(T-t)} E^Q\left[X_T^+ f\right] = E^Q\left[f S_t E^{\overline{Q}}\left[Z_T^+ \mid \mathcal{F}_t\right]\right], \tag{5.41}$$

or

$$e^{-r(T-t)} E^Q\left[X_T^+ f D_t\right] = E^Q\left[f D_t S_t E^{\overline{Q}}\left[Z_T^+ \mid \mathcal{F}_t\right]\right]. \tag{5.42}$$

But $(D_t)_{0\le t\le T}$ is a Q-martingale so the member in the right hand side of (5.42) becomes

$$E^Q\left[fD_TS_tE^{\overline{Q}}\left[Z_T^+ \mid \mathcal{F}_t\right]\right] = E^{\overline{Q}}\left[fS_tE^{\overline{Q}}\left[Z_T^+ \mid \mathcal{F}_t\right]\right] = E^{\overline{Q}}\left[fS_tZ_T^+\right]. \quad (5.43)$$

Notice that

$$\begin{cases} S_T = S_0e^{(r-\frac{\sigma^2}{2})T+\sigma W_T}, \\ S_t = S_0e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}, \end{cases} \quad (5.44)$$

which gives

$$\frac{S_T}{S_t} = e^{r(T-t)}\frac{D_T}{D_t}. \quad (5.45)$$

Thus we have

$$\begin{aligned} E^Q\left[X_T^+fD_t\right] &= E^Q\left[(\frac{X_T}{S_T})^+S_TfD_t\right] = e^{r(T-t)}E^Q\left[(\frac{X_T}{S_T})^+fS_tD_T\right] \\ &= e^{r(T-t)}E^Q\left[Z_T^+fS_tD_T\right] = e^{r(T-t)}E^{\overline{Q}}\left[Z_T^+fS_t\right], \end{aligned} \quad (5.46)$$

which proves (5.39). Thus, the right pricing PDE for Asian options is

$$\begin{cases} \frac{\partial u}{\partial t} + r(q_t - z)\frac{\partial u}{\partial z} + \frac{\sigma^2}{2}(q_t - z)^2\frac{\partial^2 u}{\partial z^2} = 0, \\ u(T, z) = z^+, \end{cases} \quad (5.47)$$

where $q_t = \mu([t, T])$. The price of the Asian option is then given in terms of u by equation (5.39). Note that the parabolic differential equation in (5.47) is the same as (4.23).

It follows from the derivation of equation (5.47) that it is enough that the equation in question holds for all $0 \le t \le T$ and all $z \in D$, where D is an open subset of $\mathbf{R}$ such that Z_t belongs to D with probability one for every $0 \le t \le T$. This may be of interest in special cases. For example, consider a European call with strike K and time of maturity T. Here $\mu = \delta_t$, $q_t = 1$ and thus

$$X_t = \int_0^t q_\lambda dS(\lambda) + \big(S(0) - K\big) = S(t) - K \quad (5.48)$$

and

$$Z_t = \frac{X(t)}{S(t)} = 1 - \frac{K}{S(t)} < 1. \quad (5.49)$$

Equation (5.47) then reads

$$\begin{cases} \frac{\partial u}{\partial t} + r(1-z)\frac{\partial u}{\partial z} + \frac{\sigma^2}{2}(1-z)^2\frac{\partial^2 u}{\partial z^2} = 0, \\ u(T,z) = z^+, \end{cases} \tag{5.50}$$

where we only have to consider z with $z < 1$. The change of variable $y = 1-z$ gives

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial y^2}, \tag{5.51}$$

and (5.50) with $z < 1$ reduces to

$$\begin{cases} \frac{\partial u}{\partial t} - ry\frac{\partial u}{\partial y} + \frac{\sigma^2}{2}y^2\frac{\partial^2 u}{\partial y^2} = 0, \\ u(T,y) = (1-y)^+, \quad y > 0. \end{cases} \tag{5.52}$$

Then according to Borell [4] we have

$$u(t,y) = E\left[\left(1 - ye^{-(r+\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}G}\right)^+\right], \tag{5.53}$$

where $G \in N(0,1)$ and it follows that

$$\begin{aligned} u(t,y) &= E\left[1 - ye^{-(r+\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}; \quad G \leq \frac{\ln\frac{1}{y} + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = D_1\right] \\ &= \Phi(D_1) - E\left[ye^{-(r+\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}; \quad G \leq D_1\right]. \end{aligned} \tag{5.54}$$

Here

$$\begin{aligned} &E\left[ye^{-(r+\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}; \quad G \leq D_1\right] \\ &= ye^{-r\tau}\int_{x\leq D_1} e^{-\frac{\sigma^2}{2}\tau+\sigma\sqrt{\tau}x-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\ &= ye^{-r\tau}\int_{x\leq D_1} e^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}}\frac{dx}{\sqrt{2\pi}} = y\Phi(D_1-\sigma\sqrt{\tau}) = y\Phi(D_2), \end{aligned} \tag{5.55}$$

where

$$D_2 = \frac{\ln\frac{1}{y} + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}. \tag{5.56}$$

Thus

$$u(t,y) = \Phi(D_1) - ye^{-r\tau}\Phi(D_2), \tag{5.57}$$

and, hence,

$$S(t)u(t,y)_{|y=\frac{K}{S(t)}} \tag{5.58}$$
$$= S_t\Phi\Big(\frac{\ln\frac{S_t}{K} + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big) - Ke^{-r\tau}\Phi\Big(\frac{\ln\frac{S_t}{K} + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big),$$

which agrees with Black-Scholes formula (3.17).

Chapter 6

An adaptive finite element method for the Asian option

Since there probably does not exist a closed form solution to the PDE (5.47), the price of the Asian option must be obtained numerically. The method used in this thesis is the finite element method as presented below.

6.1 Variational formulation

So far we have studied the pricing PDE for Asian options valid for z belonging to the whole of $\mathbf{R}$, but in order to make a numerical implementation we must limit the interval. Let Ω be an interval in $\mathbf{R}$, $\Omega = [z_0, z_J]$, and denote the boundary of Ω, that is $\{z_0, z_J\}$, by $\partial\Omega$. We define

$$H^1(\Omega) = \{v : \int_\Omega (|\nabla v|^2 + v^2)dz < \infty\} \tag{6.1}$$

and let W be the space of functions that are square integrable in time and belongs to $H^1(\Omega)$ in space, that is $W = L^2\big([0,T], H^1(\Omega)\big)$ and denote $\int_\Omega uvdz$ by (u,v). The notation $(u,v)_{\partial\Omega}$ then naturally stands for $u(z_J)v(z_J)-u(z_0)v(z_0)$. Multiplying equation (5.47) by the test function $v \in W$ and integrating over Ω and t we obtain

$$\int_0^T \Big((u_t', v) + r\big((q-z)u_z', v\big) + \frac{\sigma^2}{2}\left((q-z)^2 u_{zz}'', v\right) \Big) dt = 0. \tag{6.2}$$

Note that by integration by parts we have

$$\begin{aligned}\big((q-z)^2 u_{zz}'', v\big) = {} & \big((q-z)^2 u_z', v\big)_{\partial\Omega} + 2\big((q-z)u_z', v\big) \\ & - \big((q-z)^2 u_z', v_z'\big)\,. \end{aligned} \tag{6.3}$$

Thus equation (6.2) becomes

$$\int_0^T \Big((u'_t, v) + (r + \sigma^2)\big((q-z)u'_z, v\big) - \frac{\sigma^2}{2}\big((q-z)^2 u'_z, v'_z\big) + \frac{\sigma^2}{2}\big((q-z)^2 u'_z, v\big)_{\partial\Omega} \Big) dt = 0. \tag{6.4}$$

Introducing the artificial boundary condition $u''_{zz} = 0$ on $\partial\Omega$ (which is also used by Andreasen [1] and similar to the one used by Večeř [16]) or equivalently by equation (5.47)

$$u'_z = \frac{-u'_t}{r(q-z)} \quad \text{on} \quad \partial\Omega, \tag{6.5}$$

we get

$$\int_0^T \Big((u'_t, v) + (r + \sigma^2)\big((q-z)u'_z, v\big) - \frac{\sigma^2}{2}\big((q-z)^2 u'_z, v'_z\big) - \frac{\sigma^2}{2r}\big((q-z) u'_t, v\big)_{\partial\Omega} \Big) dt = 0. \tag{6.6}$$

We thus want to solve the following problem. Find $u \in W$ such that

$$\begin{cases} \int_0^T \big(m(u'_t, v) + a(u, v)\big) dt = 0, \\ u(T, z) = z^+, \end{cases} \tag{6.7}$$

for every $v \in W$, where

$$m(u'_t, v) = (u'_t, v) - \frac{\sigma^2}{2r}\big((q-z)u'_t, v\big)_{\partial\Omega}, \tag{6.8}$$

and

$$a(u, v) = (r + \sigma^2)\big((q-z)u'_z, v\big) - \frac{\sigma^2}{2}\big((q-z)^2 u'_z, v'_z\big). \tag{6.9}$$

6.2 Finite element approximation

A finite element approximate solution is a piecewise polynomial function that solves the variational formulation of a PDE for all test functions in an appropriate finite dimensional space (for a more general discussion about finite element theory see e.g. K. Eriksson, D. Estep, P. Hansbo and C. Johnson [7]).

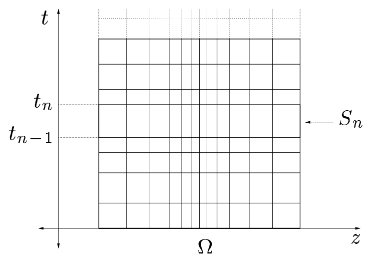

Figure 6.1: *Space-time discretization.*

We now partition $[0,T]$ as $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$, denoting each time interval by $I_n = (t_{n-1}, t_n]$ and each time step by $k_n = t_n - t_{n-1}$. Similarly we partition Ω as $z_0 < z_1 < z_2 < \cdots < z_J$, denoting the length of each interval by $h_j = z_j - z_{j-1}$.

In space, we let $P_1 \subset H^1(\Omega)$ denote the space of piecewise linear continuous functions $v(z) \in \mathbf{R}$. On each space-time "slab" $S_n = I_n \times \Omega$, we define

$$W_n^q = \{w(t,z) : w(t,z) = \sum_{j=0}^{q} t^j v_j(z), v_j \in P_1, (t,z) \in S_n\}. \tag{6.10}$$

Let $W^q \subset W$ denote the space of functions defined on $[0,T] \times \Omega$ such that $v \mid_{S_n} \in W_n^q$ for $1 \leq n \leq N$. Here we will use the continuous Galerkin method cG(1) (see e.g. Eriksson, Estep, Hansbo and Johnson [7] or D. Estep, M. Larson and R. Williams [8]) which is defined by the following discrete version of equation (6.7). Find $U \in W^1$ such that for $1 \leq n \leq N$

$$\begin{cases} \int_{I_n} \big(m(U_t', v) + a(U, v)\big) dt = 0 \quad \text{for all } v \in W_n^0 \\ U^-(t_n) = U^+(t_n), \quad n = N-1, \ldots, 1 \\ U^-(t_N) = u_T \end{cases} \tag{6.11}$$

where $U^{\pm}(t_n) = \lim_{\epsilon \to 0, \epsilon > 0} U(t_n \pm \epsilon)$. In the cG(1) method the approximation U of u is continuous piecewise linear in time and space, while the test functions v are continuous linear in space and piecewise constant in time.

6.3 Matrix equations

Using the notation $U_n = U(t_n)$ and computing the time integral in equation (6.11) yields the scheme: for $1 \leq n \leq N$

$$m(U_n - U_{n-1}, v) + k_n a\Big(\frac{U_n + U_{n-1}}{2}, v\Big) = 0 \quad \text{for all } v \in W_n^0, \tag{6.12}$$

which in fact is the classical Crank-Nicolson method.

Let the hat functions $\{\phi_j\}_{j=0}^{J}$ be the nodal basis of P_1, where only half of the first and the last hat is included (see Figure 6.2). Then $U_n \in P_1$ can be written as

$$U_n(z) = \sum_{j=0}^{J} \xi_{nj}\phi_j(z), \quad 1 \leq n \leq N, \tag{6.13}$$

and the test function v can be written as

$$v(z) = \sum_{i=0}^{J} \gamma_{ni}\phi_i(z), \quad 1 \leq n \leq N, \tag{6.14}$$

for reals $\xi_{n0}, \dots, \xi_{nJ}, \gamma_{n0}, \dots, \gamma_{nJ}$.

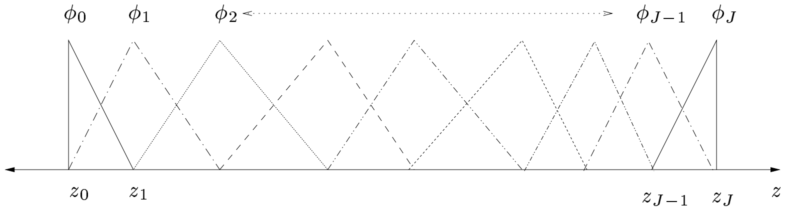

Figure 6.2: *The hat-functions ϕ.*

Let now ξ_n be the vector of all $\xi_{n,j}$, $j = 0, 1, ..., J$. If the expressions above for U and v are inserted into equation (6.12) we receive the matrix equation

$$(\xi_n - \xi_{n-1})M + (\xi_n + \xi_{n-1})\frac{k_n A}{2} = 0, \quad 1 \leq n \leq N, \tag{6.15}$$

where

$$M = (\phi_j, \phi_i) - \frac{\sigma^2}{2r}\big((q-z)\phi_j, \phi_i\big)_{\partial\Omega}, \quad 0 \leq i, j \leq J, \tag{6.16}$$

and

$$A = (r + \sigma^2)(qA_0 - A_1) - \frac{\sigma^2}{2}(q^2 A_2 - 2qA_3 + A_4), \tag{6.17}$$

where

$$\begin{aligned} &A_0 = (\phi_{j,z}, \phi_i), \quad A_1 = (z\phi_{j,z}, \phi_i), \quad A_2 = (\phi_{j,z}, \phi_{i,z}), \\ &A_3 = (z\phi_{j,z}, \phi_{i,z}), \quad \text{and} \quad A_4 = (z^2\phi_{j,z}, \phi_{i,z}), \quad 0 \leq i, j \leq J. \end{aligned} \tag{6.18}$$

Rearranging the terms in equation (6.15) we get the matrix equation we want to solve successively backwards in time in order to obtain U_0

$$\xi_{n-1} = \xi_n \left(M + \frac{k_n A}{2}\right)\left(M - \frac{k_n A}{2}\right)^{-1}, \quad 1 \leq n \leq N. \tag{6.19}$$

6.4 Error analysis based on duality

Since we are only interested in the solution in one or a few points of Ω at time $t = 0$ we want to find a good mesh that relatively fast gives an accurate solution at the points of interest. In order to find such a mesh we study the so called dual problem (see e.g Eriksson, Estep, Hansbo and Johnson [7] or Estep, Larson and Williams [8] for information about dual theory). Let $z_\alpha \in \Omega$. We now introduce the continuous dual problem to equation (5.47)

$$\begin{cases} -\psi_t' + (r+\sigma^2)\psi - (r+2\sigma^2)(q-z)\psi_z' + \frac{\sigma^2}{2}(q-z)^2\psi_{zz}'' = 0, \\ \psi(0,z) = \delta_{z_\alpha}. \end{cases} \tag{6.20}$$

For simplicity we consider this equation over the whole space interval neglecting boundary conditions. Multiplying with the error $e = u - U \in W$ and integrating over space and time we get

$$\begin{aligned} \int_0^T \Big(&- (\psi_t', e) + (r+\sigma^2)(\psi, e) \\ &- (r+2\sigma^2)\big((q-z)\psi_z', e\big) + \frac{\sigma^2}{2}\big((q-z)^2\psi_{zz}'', e\big)\Big)dt = 0. \end{aligned} \tag{6.21}$$

The functions ψ and ψ_z' are in principle zero close to $z = z_0$ and $z = z_J$ (see Figure 6.3). Using integration by parts, neglecting the boundary terms, we therefore get

$$\begin{aligned} &- (\psi(T,z), e(T,z)) + (\psi(0,z), e(0,z)) \\ &+ \int_0^T \Big((\psi, e_t') + (r+\sigma^2)(\psi, e) + (r+2\sigma^2)\big((q-z)\psi, e_z'\big)\Big)dt \\ &+ \int_0^T \Big(-(r+2\sigma^2)(\psi, e) - \frac{\sigma^2}{2}\big((q-z)^2\psi_z', e_z'\big) + \sigma^2\big((q-z)\psi_z', e\big)\Big)dt = 0. \end{aligned} \tag{6.22}$$

Note that by integration by parts we have

$$\sigma^2\big((q-z)\psi_z', e\big) = -\sigma^2\big((q-z)\psi, e_z'\big) + \sigma^2\big(\psi, e\big), \tag{6.23}$$

using this and that $\phi(0) = \delta_{z_\alpha}$ and $e(T) = 0$ we get

$$\begin{aligned} &e(0, z_\alpha) = \\ &\quad - \int_0^T \Big((\psi, e_t') + (r+\sigma^2)\big((q-z)\psi, e_z'\big) - \frac{\sigma^2}{2}\big((q-z)^2\psi_z', e_z'\big)\Big)dt. \end{aligned} \tag{6.24}$$

Considering the previous notations (equations (6.8) and (6.9)) and remembering that we can neglect the boundary terms we can also write

$$e(0, z_\alpha) = -\int_0^T \Big(m(e_t', \psi) + a(e, \psi)\Big)dt. \tag{6.25}$$

Since $e = u - U$ and u solves equation (6.7) we get

$$e(0, z_\alpha) = \int_0^T \Big(m(U_t', \psi) + a(U, \psi) \Big) dt. \tag{6.26}$$

Let $\pi : W \to W^q$ be defined by

$$\pi v I_{S_n}(t, x) = \frac{1}{k_n} \int_{I_n} Pv(t, x) dt, \tag{6.27}$$

where I_{S_n} is the indicator function of the space-time "slab" S_n and P is the node interpolant in space. Thus π is the interpolation operator such that $\pi\psi$ is piecewise linear in space and piecewise constant in time. Then since $\pi\psi$ is orthogonal to U, which can be seen from equation (6.12), equation (6.26) can be written as

$$e(0, z_\alpha) = \int_0^T \Big(m(U_t', \psi - \pi\psi) + a(U, \psi - \pi\psi) \Big) dt. \tag{6.28}$$

If we solve the dual problem numerically for ψ and also calculate $\psi - \pi\psi$ we can see where we have to use a fine mesh. As seen from the Figures 6.3

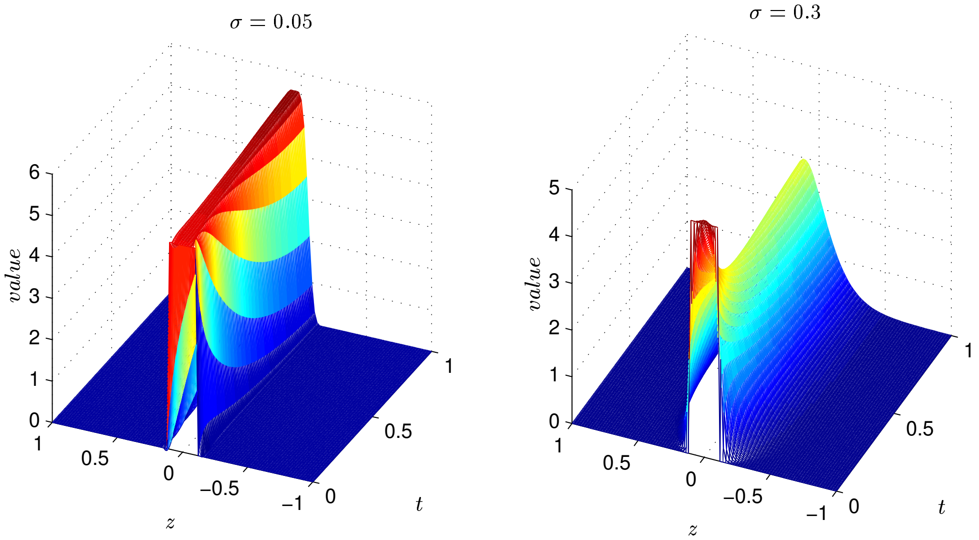

Figure 6.3: *ψ for two different values of σ when r=0.10. Computed with 100 space points and 200 time points, using the boundary condition $\psi(0, z) = 5I_{[-0.1,0.1]}$.*

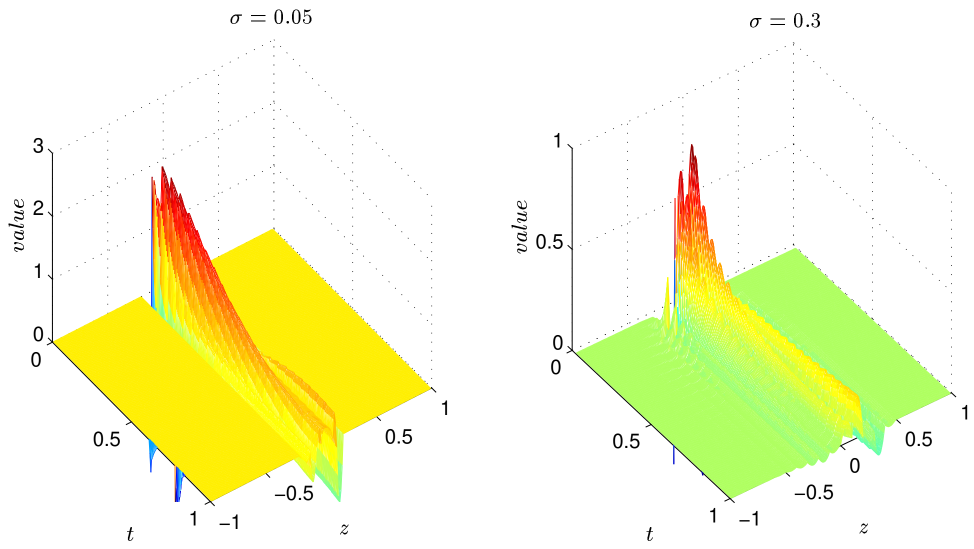

Figure 6.4: $\psi - \pi\psi$ *for two different values of* σ *when r=0.10.* ψ *computed with 200 space points and 200 time points and* $\pi\psi$ *computed with 20 space points and 20 time points. The boundary condition used was* $\psi(0, z) = 5I_{[-0.1,0.1]}$.

and 6.4 the solution to the dual problem differs from zero only within a short interval of Ω. Denote this interval by $\omega = [-z_b, z_b] \in \Omega$, note that z_b depends on the value of the volatility σ. This means that we may use a more sparse mesh outside ω and thus save computation time. In Figure 6.4 we also see that the solution is bigger near time $t = 0$, implying that one perhaps should use a finer time step there.

6.5 Results

The implementation was done in MATLAB on a SUN sparc station. Many different meshes were used, both with constant time and space step and varying. A very fine mesh was used to compute what is regarded as the "exact solution".

As noted in the previous section we only need to use a fine mesh at the centre of Ω. We will later see how this fact dramatically improves the speed of the numerical computation. In the following we will use $z_0 = -1$ and $z_J = 1$, the accuracy is not improved if a larger interval is used.

Table 6.1 compares values of the European call calculated using the cG1 finite element method mentioned above with the analytical value derived by

Black-Scholes (see Chapter 3, Theorem 3). We see that the FEM method is very stable and has a maximum relative error of 0.06 percent when 400 time points are used.

σ	K	FE(200)	FE(400)	Black-Scholes	Relative error (%)
	90	14.6207	14.6268	14.6288	0.0137
0.10	100	6.7972	6.8030	6.8050	0.0294
	110	2.1687	2.1726	2.1739	0.0598
	90	16.6983	16.6981	16.6994	0.0078
0.20	100	10.4468	10.4496	10.4506	0.0096
	110	6.0375	6.0395	6.0401	0.0099
	90	19.6932	19.6965	19.6974	0.0046
0.30	100	14.2273	14.2304	14.2313	0.0063
	110	10.0148	10.0189	10.0201	0.0120

Table 6.1: *The European call calculated using the FEM method with 200 and 400 time points compared to Black-Scholes analytical value when r=0.05, T=1 and t=0. The relative error is between the FE(400) solution and the analytical solution.*

In Figure 6.5 we see the average rate call option value calculated using a uniform mesh. Table 6.2 compares the results of the method developed in this paper with the results of Večeř [16], Zvan, Forsyth and Vetzal [17] and Rogers and Shi [14]. To be consistent with their results a uniform mesh with same number of time and space points (200 space points and 400 time points) was used in the computation of the finite element results in Table 6.2. The Monte Carlo results were obtained from Večeř [16] and the lower and upper bounds are from Rogers and Shi [14]. As seen from the table all methods are accurate and always give answers within analytical bounds. The most important difference between them is the computation time required to receive the results.

If we instead use a mesh that is finer at the centre of Ω we will improve the execution time. As mentioned in the previous section we only need to use a fine mesh in the space interval $\omega = [-z_b, z_b] \in \Omega$, where z_b depends on the value of the volatility σ. Here we use a mesh were $z_b = 0.2$ for $\sigma = 0.05 - 0.1$, $z_b = 0.3$ for $\sigma = 0.2$ and $z_b = 0.4$ for $\sigma = 0.3$. Inside ω we use a uniform mesh were the length of each space interval h_j is half the length of the time step and outside ω we use a mesh were the length of the space intervals doubles each step towards the boundary $\partial\Omega$ (see Figure 6.6). We then improve the execution time without affecting the accuracy.

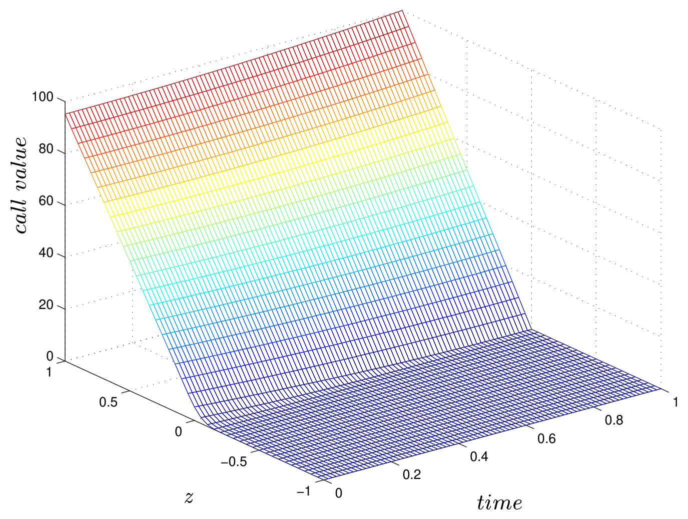

Figure 6.5: *An average rate call option with r=0.10,* $\sigma = 0.10$*, T=1 and t=0.*

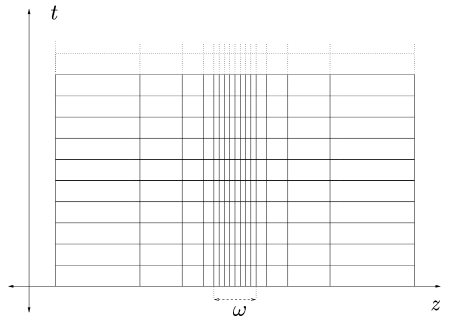

Figure 6.6: Space-time discretization.

Table 6.3 gives some results for the average rate call option computed with the non-uniform mesh mentioned above. The lower and upper bounds are from Rogers and Shi [14]. The execution time of the program with 400 time points is now just a few seconds and at least 5 times faster than if a uniform mesh is used. These execution times are very promising considering that the program was written in MATLAB, the same program written in C will surely have execution times in less than a second.

How much can be gained in accuracy by refining the time steps close to time $t = 0$ as mentioned above? Table 6.4 gives the values of an average rate call option computed with a new mesh were the last G number of time steps are successively decreased with a factor δ, were the last G number of time

σ	K	Foufas	Večeř	Zvan et al.	Monte Carlo	Lower	Upper
	95	11.112	11.112	11.094	11.094	11.094	11.114
0.05	100	6.810	6.810	6.793	6.795	6.794	6.810
	105	2.754	2.750	2.748	2.745	2.744	2.761
	90	15.416	15.416	15.399	15.399	15.399	15.445
0.10	100	7.042	7.036	7.030	7.028	7.028	7.066
	110	1.422	1.421	1.410	1.418	1.413	1.451
	90	15.659	15.659	15.643	15.642	15.641	15.748
0.20	100	8.427	8.424	8.409	8.409	8.408	8.515
	110	3.570	3.568	3.554	3.556	3.554	3.661
	90	16.533	16.533	16.514	16.516	16.512	16.732
0.30	100	10.231	10.230	10.210	10.210	10.208	10.429
	110	5.750	5.748	5.729	5.731	5.728	5.948

Table 6.2: *Comparison of results of different methods for the average rate call with r=0.15, $S_0 = 100$, T=1 and t=0. The Monte Carlo results are from Večeř [16] and the lower and upper bounds are from Rogers and Shi [14].*

steps refers to the time steps that lie in the interval $[0, t_1]$, $t_1 < T$, since we are solving backwards in time towards time $t = 0$. After some testing it was shown that in order to improve the accuracy it works well if the factor δ is close to one. We see that compared to the results of the previous method with constant time step given in Table 6.3, this new mesh leads to a bit more accurate answers. The mean relative error is now 0.520 percent instead of 0.530 percent as before, thus the increase in accuracy is very small and since the execution time now is greater we conclude that one probably should use the same time step everywhere.

Figure 6.8 shows how the value of the ordinary European call and the average rate call depends very much on the value of the volatility σ. We see that the average rate call is less dependent on σ than the European call and again we see that the average rate call is cheaper than the European call in agreement with Theorem 6 in Chapter 3.

Table 6.5 gives the value of the discrete average rate call for various strikes. The results of the finite element method given in this paper are compared with results of the finite difference method and Monte Carlo results given in Andreasen [1]. As can be seen they are in excellent agreement. We also see that the maximum relative error between the solution computed with 500 time points and the "exact" solution is about 0.02 percent, and that the finite element method gives accurate answers already for a small number of time steps.

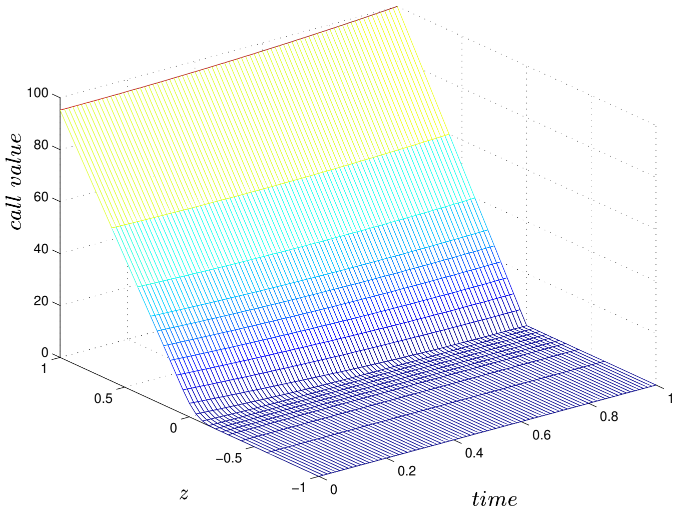

Figure 6.7: *An average rate call option with r=0.10, $\sigma = 0.10$, T=1 and t=0. Computed on a non-uniform mesh with 40 space points and 80 time points.*

σ	K	FE(80)	FE(200)	FE(400)	Exact	R. E. (%)
	90	13.762	13.751	13.739	13.728	0.082
0.05	100	4.548	4.728	4.731	4.725	0.127
	110	0.016	0.063	0.075	0.078	4.215
	90	13.778	13.754	13.744	13.733	0.078
0.10	100	5.225	5.265	5.264	5.255	0.160
	110	0.708	0.733	0.734	0.731	0.438
	90	14.158	14.158	14.150	14.138	0.081
0.20	100	7.078	7.066	7.055	7.042	0.186
	110	2.735	2.720	2.711	2.701	0.385
	90	15.291	15.270	15.257	15.242	0.100
0.30	100	9.124	9.089	9.073	9.056	0.189
	110	4.930	4.892	4.877	4.862	0.315

Table 6.3: *An average rate call with r=0.10, $S_0 = 100$, T=1 and t=0, computed using a non uniform mesh. FE refers to the finite element solution with the number of time steps given inside the parenthesis and exact refers to the "exact solution" that was computed using a uniform mesh with 4000 space points and 8000 time points. R.E. refers to the relative error between the exact solution and FE(400).*

σ	K	FT(213), G=35	FT(432), G=60	Exact	R. E. (%)
	90	13.750	13.739	13.728	0.080
0.05	100	4.727	4.731	4.725	0.127
	110	0.063	0.075	0.078	4.215
	90	13.753	13.743	13.733	0.075
0.10	100	5.264	5.263	5.255	0.150
	110	0.732	0.734	0.731	0.397
	90	14.157	14.149	14.138	0.077
0.20	100	7.064	7.054	7.042	0.178
	110	2.719	2.710	2.701	0.367
	90	15.268	15.256	15.242	0.095
0.30	100	9.087	9.072	9.056	0.180
	110	4.891	4.877	4.862	0.300

Table 6.4: *The average rate call with r=0.10,* $S_0 = 100$*, T=1 and t=0. FT refers to the solution computed with a non uniform mesh were the last G number of time steps are successively decreased with a factor* $\delta = 0.97$*, were the last G number of time steps refers to the time steps that lie in the interval* $[0, t_1]$*,* $t_1 < T$*, since we are solving backwards in time towards time* $t = 0$*. The total number of time steps is given inside parenthesis. Exact refers to the "exact solution" that was computed using a uniform mesh with 4000 space points and 8000 time points. R.E. refers to the relative error between the exact solution and FT(432).*

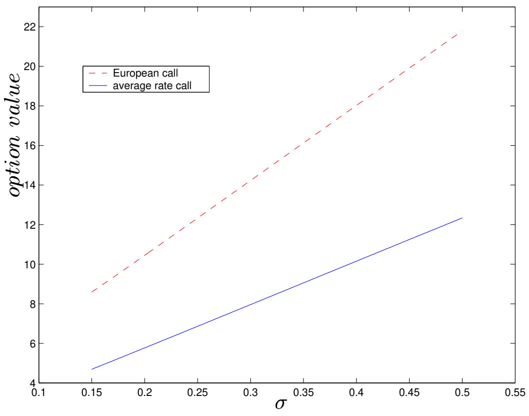

Figure 6.8: *The average rate call compared to the ordinary European call for different* σ *when* $S_0 = 100$*, K=100, r=0.05, T=1 and t=0.*

K	FE(100)	FE(200)	FE(500)	FD(500)	MC	Exact	R. E. (%)
90.0	12.97	12.98	12.98	12.99	12.98	12.9853	0.0062
92.5	11.04	11.05	11.05	11.05	11.05	11.0504	0.0072
95.0	9.26	9.27	9.27	9.27	9.27	9.2690	0.0076
97.5	7.65	7.66	7.66	7.66	7.67	7.6597	0.0078
100.0	6.23	6.23	6.23	6.23	6.24	6.2345	0.0080
102.5	4.99	5.00	5.00	5.00	5.01	4.9975	0.0080
105.0	3.94	3.94	3.95	3.95	3.96	3.9455	0.0101
107.5	3.06	3.07	3.07	3.07	3.08	3.0685	0.0130
110.0	2.35	2.35	2.35	2.35	2.36	2.3516	0.0170

Table 6.5: *A discrete average rate call with* $S_0 = 100$, $\sigma = 0.2$, *r=0.05, T=1, t=0, N=10 and* $t_k = 0.1k$. *FE refers to the finite element solution computed with a nonuniform mesh with the number of time steps given inside the parenthesis. FD refers to the finite difference solution and MC refers to the Monte Carlo solution based on* 10^5 *simulations with a control variate technique, both given in Andreasen [1]. Exact refers to the "exact solution" that was computed using a uniform mesh with 4000 space points and 8000 time points. R.E. refers to the relative error between the exact solution and FE(500).*

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