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A DISCONTINUOUS GALERKIN METHOD FOR THE PLATE EQUATION

PETER HANSBO AND MATS G. LARSON

Abstract. We present a discontinuous Galerkin method for the plate problem. The method employs a discontinuous approximation space allowing, non matching grids and different types of approximation spaces. Continuity is enforced weakly through the variational form. Discrete approximations of the normal and twisting moments and the transversal force, which satisfy the equilibrium condition on an element level, occur naturally in the method. We show optimal a priori error estimates in various norms and investigate locking phenomena when certain stabilization parameters tend to infinity. Finally, we relate the method to two classical elements; the nonconforming Morley element and the C^1 Argyris element.

1. Introduction

In this paper we propose and analyze a discontinuous Galerkin (dG) method for the plate equation describing the transversal deflection of a thin plate under a transversal load. The method is based on the classical method first proposed by Nitsche in the context of weak enforcement of boundary conditions [14] and later extended to a discontinuous method with weak enforcement of the continuity of the solution at interior edges by Douglas and Dupont [8], Baker [3], Wheeler [15], and Arnold [2]. In the last few years there has been a renewed interest in these methods, see for instance the proceedings [7] for a comprehensive overview of recent work.

The use of discontinuous approximation spaces lead to several advantages, for instance, one can use different types of approximation spaces on different elements without enforcing continuity; non matching grids, see Becker and Hansbo [4], can also be used. Using the added richness of the spaces one can also construct locking free schemes for nearly incompressible linear elastic materials, see Hansbo and Larson, [9]. Further, discontinuous methods enjoy a local elementwise conservation property, a property often desired in applications. The obvious disadvantage of the dG method is the increased number of degrees

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of freedom for the same order of approximation, see [10], for a comparison of the number of degrees of freedom in the continuos and discontinuous Galerkin methods.

It is well known that so called twisting moments occur at corners on the boundary in the plate equation. When formulating a discontinuous Galerkin method for the plate equation these twisting moments occur at each node in the triangulation, and one has to discretize them in a proper way. Our work extends the earlier work of Baker [3], where a method, similar to ours, for the biharmonic equation is presented. In this case the twisting moments do not occur.

Discrete approximations of the bending and twisting moments, and the transversal force, of optimal order and such that elementwise equilibrium is satisfied, arise in a natural way in the method.

As is well known in the symmetric dG method one needs to include terms (which vanish for the exact solution) that stabilize (or penalize) discontinuities and choose corresponding parameters sufficiently large in order to obtain a coercive bilinear form. It is easy to see that it is enough to stabilize only certain projections of the discontinuities. This observation allows us to relate our method to the classical nonconforming element of Morley. Here second degree, polynomials are used, and the nodal values and the normal derivatives at the midpoint of each edge are chosen as degrees of freedom. These methods are obtained by choosing minimal stabilization of discontinuities and letting the stabilization parameters tend to infinity. Further, in a similar fashion, we obtain the Argyris C^1 element for fifth order polynomials on triangles. See, also Larson and Niklasson [11] for related investigations in the case of the Poisson equation.

We believe the dG method is particularly suitable for two dimensional formulation in structural mechanics, such as plates and shells, where typically complicated elements are used.

The remainder of the paper is organized as follows: in Section 2 we introduce the plate equation and the dG method; in Section 3 we prove error estimates; and in Section 4 we present a numerical example.

2. The dG method

Consider a thin elastic plate with center surface represented by a domain $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$. We seek the transversal deflection u when a transversal load f acts on the plate, and various physical boundary conditions are used at the boundary. Assuming small deflections and a linearly elastic material the constitutive equation reads

$$\sigma_{ij} = \lambda \Delta u \delta_{ij} + \mu \kappa_{ij}(u), \quad i,j = 1,2, \tag{2.1}$$

where σ_{ij} are the moments, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ Δ is the Laplacian, and

$$\kappa_{ij}(u) = u_{,ij}, \tag{2.2}$$

defines the curvature tensor. Furthermore, with t the thickness of the plate, E Young's modulus, ν Poisson's ratio, and

$$D = \frac{Et^3}{12(1-\nu^2)}$$

the flexural rigidity of the plate, we have that $\lambda = D\nu$ and $\mu = D(1-\nu)$. Here, and below, we adopt the simplified notation $u_{,i}$ for the partial derivative $\partial u/\partial x_i$. Further, with f a distributed normal load, we have the equilibrium equation

$$\sigma_{ij,ij} = f, \tag{2.3}$$

where we used the summation convention.

To define the boundary conditions we introduce some notation. Let $n = (n_1, n_2)$ be the outward unit normal, and $t = (n_2, -n_1)$ the associated tangent, of $\partial\Omega$. We will need the following quantities

$$u_{,n} = u_{,j} n_j, \tag{2.4}$$

$$u_{,t} = u_{,j} t_j, \tag{2.5}$$

$$M_{nn} = \sigma_{ij} n_i n_j, \tag{2.6}$$

$$M_{nt} = \sigma_{ij} n_i t_j, \tag{2.7}$$

$$T = \sigma_{ij,j} n_i + M_{nt,t} \tag{2.8}$$

where $u_{,n}$ and $u_{,t}$ are the normal and tangential derivatives, M_{nn} and M_{nt} are the normal and twisting moments, and T is the transversal force, at the boundary. We may clearly consider M_{nn}, M_{nt}, and T, as functions of σ or of u, since σ is a function of u.

We consider three types of boundary conditions modeling a clamped, simply supported, and free boundary. Splitting the boundary into three corresponding disjoint parts $\partial\Omega = \partial\Omega_C \cup \partial\Omega_S \cup \partial\Omega_F$ the three boundary conditions read

$$u = u_{,n} = 0 \qquad \text{on } \partial\Omega_C, \tag{2.9}$$

$$u = M_{nn} = 0 \qquad \text{on } \partial\Omega_S, \tag{2.10}$$

$$M_{nn} = T = 0 \qquad \text{on } \partial\Omega_F. \tag{2.11}$$

2.1. **The mesh and discontinuous spaces.** Let $\mathcal{K} = \{K\}$ be a partition of Ω into shape regular elements, called the mesh. For simplicity, we assume in our a priori error analysis that the mesh is quasi uniform with meshsize h, see [6].

The set of edges in the mesh is denoted by $\mathcal{E} = \{E\}$ and we split $\mathcal{E}$ into four disjoint subsets

$$\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_C \cup \mathcal{E}_S \cup \mathcal{E}_F, \tag{2.12}$$

where $\mathcal{E}_I$ is the set of edges in the interior of Ω, and $\mathcal{E}_C$, $\mathcal{E}_S$, $\mathcal{E}_F$, are the sets of edges on the clamped, simply supported, and free parts of the boundary, respectively.

To each edge we associate a fixed unit normal vector

$$n_E = (n_{E,1}, n_{E,2}),$$

such that on the boundary $\partial\Omega$, n_E is the exterior unit normal, and a fixed tangent vector $t_E = (n_{E,2}, -n_{E,1})$. Further to each node in ∂E we associate a normal $n_{\partial E} = 1$ if t_E is directed out of E and $n_{\partial E} = -1$ if t_E is directed into E. We also use the notation $n_{\partial K}$ for the exterior normal of an element K. For simplicity, we usually use the notation $n = n_E$ and $t = t_E$ in an integral over the edge E, and similarly for integrals over ∂K and ∂E.

Let $\mathcal{W}$ be the space of discontinuous piecewise polynomials of degree p defined on $\mathcal{K}$, i.e.,

$$\mathcal{W} = \bigoplus_{K \in \mathcal{K}} \mathcal{P}_p(K), \tag{2.13}$$

where $\mathcal{P}_p(K)$ denote the space of polynomials of degree p. In general, the degree of polynomials may change from element to element or other spaces, such as spectral elements, may be used on some elements. For simplicity we confine our attention to polynomials of fixed degree.

For convenience we also define the space

$$\mathcal{V} = \mathcal{W} + H^4(\Omega). \tag{2.14}$$

Here and below $H^s(\omega)$ denotes the standard Sobolev space of order s on the set $\omega \subset \Omega$, with norm $\|\cdot\|_{s,\omega}$.

2.2. **Variational formulation on an element.** To motivate the definition of the dG method we begin with deriving a Green's formula on an element $K \in \mathcal{K}$. Starting from the equilibrium equation (2.3), multiplying with a test function v, and using repeated partial integration give

$$\begin{aligned}(\sigma_{ij,ij}, v)_K &= -(\sigma_{ij,i}, v_{,j})_K + (\sigma_{ij,i}, v n_j)_{\partial K} \\ &= (\sigma_{ij}, v_{,ij})_K - (\sigma_{ij} n_i, v_{,j})_{\partial K} + (\sigma_{ij,i}, v n_j)_{\partial K} \\ &= (\sigma_{ij}, v_{,ij})_K - (M_{nn}, v_{,n})_{\partial K} \\ &\quad - (M_{nt}, v_{,t})_{\partial K} + (\sigma_{ij,i}, v n_j)_{\partial K},\end{aligned} \tag{2.15}$$

where in the last equality we used the identity

$$v_{,j} = v_{,n} n_j + v_{,t} t_j. \tag{2.16}$$

To continue we note that the boundary ∂K of an element K consists of a number of smooth edges $E \subset \partial K$, connected at the nodes. On each edge we have using partial integration

$$(M_{nt}, v_{,t})_E = -(M_{nt,t}, v)_E + (M_{nt}, v n_{\partial E})_{\partial E}. \tag{2.17}$$

Combining (2.15) and (2.17) with the constitutive equation (2.1) and the equilibrium equation (2.3) we get the following variational statement on each element

$$\begin{aligned}&\lambda(\Delta u, \Delta v)_K + \mu(\kappa_{ij}(u), \kappa_{ij}(v))_K \\ &\qquad = \sum_{E \subset \partial K} \Big((M_{nn}, v_{,n})_E - (T, v)_E + (M_{nt}, v n_{\partial E})_{\partial E} \Big) + (f, v)_K,\end{aligned} \tag{2.18}$$

for all $v \in H^4(K)$. Note, in particular, the presence of the pointwise twisting moments arising at the corners. These contributions are an effect of the presence of the curvature tensor in the constitutive equation (2.1) and does not occur for the biharmonic equation. Note also that the twisting moments vanish if the boundary is smooth.

2.3. **Discrete moments and transversal force.** We now wish to extend our elementwise variational statement (2.18) to a variational statement on $\mathcal{V}$. We then need to define the moments and transversal force, $M_{nn}(v)$, $M_{nt}(v)$ and $T(v)$, on each edge $E \in \mathcal{E}$ for functions $v \in \mathcal{W}$, which are discontinuous at edges. Motivated by the stability analysis presented below we introduce the following definitions

$$M_{nn}(v) = \langle M_{nn}(v) \rangle - \beta_1 h^{-1} P_{l_1}[v_{,n}], \tag{2.19}$$

$$T(v) = \langle T(v) \rangle + \beta_2 h^{-3} P_{l_2}[v], \tag{2.20}$$

$$M_{nt}(v) = \langle M_{nt}(v) \rangle - \beta_3 h^{-2} n_{\partial E}[v], \tag{2.21}$$

on each edge $E \in \mathcal{E}$. Here β_1, β_2, and β_3 are positive parameters and P_{l_i} denotes the L^2 projection onto polynomials $\mathcal{P}_{l_i}(E)$ of degree l_i defined on the edge E, with $p-2 \le l_1 \le p$ and $p-3 \le l_2 \le p$. Further we employed the following notation for the average

$$\langle v \rangle = \begin{cases} (v^+ + v^-)/2 & E \in \mathcal{E}_I, \\ v^+ & E \in \mathcal{E} \setminus \mathcal{E}_I, \end{cases} \tag{2.22}$$

and jump

$$[v] = \begin{cases} v^+ - v^- & E \in \mathcal{E}_I, \\ v^+ & E \in \mathcal{E} \setminus \mathcal{E}_I, \end{cases} \tag{2.23}$$

of a function $v \in \mathcal{W}$ at an edge, where $v^\pm(x) = \lim_{s \to 0^+} v(x \mp s n_E)$. Inserting these definitions of the moments and transversal force on the edges into (2.18) gives an elementwise variational statement in $\mathcal{W}$. This equation is however nonsymmetric, but we can easily symmetrize it without affecting consistency, since $[u] = 0$ for $E \in \mathcal{E}_I \cup \mathcal{E}_C \cup \mathcal{E}_S$, and $[u_{,n}] = 0$ for $E \in \mathcal{E}_I \cup \mathcal{E}_C$ for the exact solution. In the next section we give the global statement of the resulting method.

2.4. **The dG method.** The dG method for the plate equation is defined by: find $U \in \mathcal{W}$ such that

$$a(U, v) = l(v) \quad \text{for all } v \in \mathcal{W}, \tag{2.24}$$

where the bilinear form is defined by

$$
\begin{aligned}
a(v,w) = \sum_{K\in\mathcal{K}} & \lambda(\Delta v, \Delta w)_K + \mu(\kappa_{ij}(v), \kappa_{ij}(w))_K \\
& - \sum_{E\in\mathcal{E}\setminus(\mathcal{E}_S\cup\mathcal{E}_F)} \Big((\langle M_{nn}(v)\rangle, [w_{,n}])_E + ([v_{,n}], \langle M_{nn}(w)\rangle)_E \Big) \\
& \qquad + \beta_1 (h^{-1} P_{l_1}[v_{,n}], P_{l_1}[w_{,n}])_E \\
& + \sum_{E\in\mathcal{E}\setminus\mathcal{E}_F} \Big((\langle T(v)\rangle, [w])_E + ([v], \langle T(w)\rangle)_E \Big) \\
& \qquad + \beta_2 (h^{-3} P_{l_2}[v], P_{l_2}[w])_E \\
& - \sum_{E\in\mathcal{E}} \Big((\langle M_{nt}(v)\rangle, n_{\partial E}[w])_{\partial E} + (n_{\partial E}[v], \langle M_{nt}(w)\rangle)_{\partial E} \Big) \\
& \qquad + \beta_3 (h^{-2}[v], [w])_{\partial E},
\end{aligned}
\tag{2.25}
$$

with positive real parameters β_i, $i = 1, 2, 3$, and the linear functional by

$$
l(w) = (f, w). \tag{2.26}
$$

2.5. **Elementwise equilibrium.** We note that taking $v|_K \in \mathcal{P}_1(K)$ and $v|_{\Omega\setminus K} = 0$ in (2.24) we obtain the following elementwise equilibrium condition for the discrete moment and transversal force

$$
\sum_{E\subset\partial K} \Big((M_{nn}, v_{,n})_E - (T, v)_E + (M_{nt}, v n_{\partial E})_{\partial E} \Big) + (f, v)_K = 0, \tag{2.27}
$$

for all $v \in \mathcal{P}_1(K)$. This means that with the discrete moments and transversal force each element is in equilibrium.

3. Error estimates

3.1. **The energy norm.** We equip $\mathcal{W}$ with the following energy norm

$$
\begin{aligned}
|||v|||^2 = {} & |||v|||^2_{\mathcal{K}} + \|\langle M_{nn}(v)\rangle\|^2_{\mathcal{E}\setminus(\mathcal{E}_S\cup\mathcal{E}_F)} \\
& + \|h\langle T(v)\rangle\|^2_{\mathcal{E}\setminus\mathcal{E}_F} + \|\langle M_{nt}(v)\rangle\|^2_{\partial\mathcal{E}} \\
& + \|h^{-1} P_{l_1}[v_{,n}]\|^2_{\mathcal{E}\setminus(\mathcal{E}_S\cup\mathcal{E}_F)} + \|h^{-2} P_{l_2}[v]\|^2_{\mathcal{E}\setminus\mathcal{E}_F} + \|h^{-2}[v]\|^2_{\partial\mathcal{E}}.
\end{aligned}
\tag{3.1}
$$

Here and below we employed the following notations

$$
|||w|||^2_{\mathcal{K}} = \sum_{K\in\mathcal{K}} \lambda(\Delta w, \Delta w)_K + \mu(\kappa_{ij}(w), \kappa_{ij}(w))_K, \tag{3.2}
$$

$$
\|w\|^2_{\mathcal{F}} = \sum_{E\in\mathcal{F}} \|h^{1/2} w\|^2_E, \tag{3.3}
$$

$$
\|w\|^2_{\partial\mathcal{F}} = \sum_{E\in\mathcal{F}} \sum_{x\in\partial E} h^2 w(x)^2, \tag{3.4}
$$

for any subset $\mathcal{F} \subset \mathcal{E}$ of edges. Note, that $|||\cdot|||^2$ is a norm on $\mathcal{V}$ since if $|||v|||_{\mathcal{K}} = 0$, then v must be a discontinuous piecewise linear function, and if also $\|h^{-2}[v]\|^2_{\partial\mathcal{E}} = 0$ then v is also continuous and zero on $\partial\Omega$. Finally, if $\|h^{-1}P_{l_1}[v_{,n}]\|^2_{\mathcal{E}_1} = 0$ then $v = 0$, since $[v_{,n}]$ is a constant function on each edge, so that $P_{l_1}[v_{,n}] = [v_{,n}]$.

Furthermore, we shall need the following inverse estimates.

Lemma 3.1. *With the above definitions of the norms we have*

$$\|\langle M_{nn}(v)\rangle\|^2_{\mathcal{E}\setminus(\mathcal{E}_S\cup\mathcal{E}_F)} \leq C_1|||v|||^2_{\mathcal{K}}, \tag{3.5}$$

$$\|h\langle T(v)\rangle\|^2_{\mathcal{E}\setminus\mathcal{E}_F} \leq C_2|||v|||^2_{\mathcal{K}}, \tag{3.6}$$

$$\|\langle M_{nt}(v)\rangle\|^2_{\partial\mathcal{E}} \leq C_3|||v|||^2_{\mathcal{K}}, \tag{3.7}$$

for all $v \in \mathcal{W}$. Here C_i denote constants independent of the meshsize h and the parameters β_i.

Proof. We show these estimates on each element and obtain the global estimates by summing over the elements. First we map the element to a unit size reference element and conclude that the corresponding estimate holds by using finite dimensionality together with the observation that if the right hand side is zero v must linear function on the element and thus the left hand side is zero in all three cases. Finally, mapping back to the original element yields the desired estimates. ☐

Lemma 3.2. *We have the following two trace inequalities*

$$\|v\|^2_{\mathcal{E}} \leq C\sum_{K\in\mathcal{K}}\Big(\|v\|^2_K + h^2\|v\|^2_{1,K}\Big), \tag{3.8}$$

$$\|v\|^2_{\partial\mathcal{E}} \leq C\sum_{K\in\mathcal{K}}\Big(\|v\|^2_K + h^2\|v\|^2_{1,K} + h^4\|v\|^2_{2,K}\Big), \tag{3.9}$$

for all $v \in \mathcal{V}$.

Proof. We first recall that on each edge E or element K, we have the well known trace inequality

$$\|v\|^2_{\partial S} \leq C\Big(h^{-1}\|v\|^2_S + h\|v\|^2_{1,S}\Big), \tag{3.10}$$

with $S = E$ or $S = K$. We prove (3.10) by mapping to a unit size reference edge, or reference element, S_{ref} invoking the trace inequality $\|v\|^2_{\partial S_{ref}} \leq C\|v\|_{S_{ref}}\|v\|_{1,S_{ref}}$, see [6], and, finally, mapping back to S. Now to prove (3.8) we start from the definition of $\|\cdot\|_{\mathcal{E}}$ and use (3.10) elementwise to get

$$\|v\|^2_{\mathcal{E}} = \sum_{E\in\mathcal{E}} h\|v\|^2_E \leq C\sum_{K\in\mathcal{K}}\Big(\|v\|^2_K + h^2\|v\|^2_{1,K}\Big). \tag{3.11}$$

Next for the second statement (3.9) employing (3.10) twice, first edgewise and then elementwise give

$$
\begin{aligned}
\|v\|_{\partial\mathcal{E}}^2 &= \sum_{E\in\mathcal{E}} h^2 \|v\|_{\partial E}^2 \\
&\leq C \sum_{E\in\mathcal{E}} \Big(h\|v\|_E^2 + h^3 \|v\|_{1,E}^2 \Big) \\
&\leq C \sum_{K\in\mathcal{K}} \Big(\|v\|_K^2 + h^2\|v\|_{1,K}^2 + h^4 \|v\|_{2,K}^2 \Big), \qquad (3.12)
\end{aligned}
$$

and thus the proof is complete. □

Lemma 3.3. *There is an interpolation operator $\pi : \mathcal{V} \to \mathcal{W}$ such that*

$$
|||u - \pi u||| \leq C \begin{cases} h\|u\|_3 + h^2\|u\|_4 & p = 2, \\ h^{p-1}\|u\|_{p+1} & p \geq 3, \end{cases} \qquad (3.13)
$$

for all sufficiently smooth $u \in \mathcal{V}$.

Proof. We may define the interpolant π to be the usual nodal Lagrange interpolant since all $v \in \mathcal{V}$ are in fact continuous. We prove the error estimate by estimating the energy norm $|||v|||$ as follows

$$
|||v|||^2 \leq C \sum_{K\in\mathcal{K}} \sum_{j=0}^{3} h^{-2(j-2)} \Big(\|v\|_{j,K}^2 + h^2 \|v\|_{j+1,K}^2 \Big), \qquad (3.14)
$$

and then setting $v = u - \pi u$ and employing the standard interpolation error estimate

$$
\|u - \pi u\|_{s,K} \leq C h^{p+1-s} \|u\|_{p+1,K}, \qquad (3.15)
$$

with $s = 0, 1, 2, 3$. To prove (3.14) we first note that

$$
|||v|||_{\mathcal{K}}^2 \leq C \sum_{K\in\mathcal{K}} \|v\|_{2,K}^2, \qquad (3.16)
$$

for each $K \in \mathcal{K}$. Next to estimate the contributions from the edges we invoke (3.8), and the pointwise contributions are zero, due to the choice of interpolant. □

3.2. Main results.

Lemma 3.4. *Here we collect three basic results on consistency, continuity, and coercivity:*
1. With u the exact solution of the plate equation and U the approximate dG solution defined by (2.24) we have

$$
a(u - U, v) = 0 \quad \textit{for all } v \in \mathcal{W}. \qquad (3.17)
$$

2. There is a constant C, which is independent of h but in general depends on β, such that

$$
a(u - \pi u, v) \leq C|||u - \pi u|||\;|||v||| \quad u \in \mathcal{V}, v \in \mathcal{V}. \qquad (3.18)
$$

3. For β sufficiently large the coercivity estimate

$$c|||v|||^2 \leq a(v,v) \quad v \in \mathcal{W}, \tag{3.19}$$

holds, with a positive constant c independent of h and β.

Remark 3.1. *If there is an interpolation operator $\pi : \mathcal{V} \to \mathcal{W}$, satisfying (3.13), such that*

$$\|P_{l_1}[\pi u_{,n}]\|_{\mathcal{E}} = 0, \tag{3.20}$$

$$\|P_{l_2}[\pi u]\|_{\mathcal{E}} = 0, \tag{3.21}$$

$$\|[\pi u]\|_{\partial\mathcal{E}} = 0, \tag{3.22}$$

then the constant C in (3.18) is also independent of β. In fact, in the limit $\beta \to \infty$ the continuity requirements (3.20–3.22) are enforced strongly. Typically, enforcing (3.20–3.22) restricts the choice of interpolant, which may lead to a loss of accuracy.

Proof. 1. This fact is a direct consequence of the fact that the exact solution u satisfies the variational statement (2.24).
2. This estimate follows immediately using the Cauchy Schwarz inequality, from the definitions of the bilinear form (2.25) and the energy norm (3.1).
3. We have

$$\begin{aligned} a(v,v) = {} & |||v|||^2_{\mathcal{K}} \\ & - \sum_{E \in \mathcal{E} \setminus (\mathcal{E}_S \cup \mathcal{E}_F)} 2(\langle M_{nn}(v) \rangle, [v_{,n}])_E + \beta_1 \|h^{-1} P_{l_1}[v_{,n}]\|^2_{\mathcal{E} \setminus (\mathcal{E}_S \cup \mathcal{E}_F)} \\ & - \sum_{E \in \mathcal{E} \setminus \mathcal{E}_F} 2(\langle T(v) \rangle, [v])_E + \beta_2 \|h^{-2} P_{l_2}[v]\|^2_{\mathcal{E} \setminus \mathcal{E}_F} \\ & - \sum_{E \in \mathcal{E}} 2(\langle M_{nt}(v) \rangle, n_{\partial E}[v])_{\partial E} + \beta_3 \|h^{-1}[v]\|^2_{\partial\mathcal{E}}. \end{aligned}$$

Recalling the definition of the L^2 projections P_{l_i}, $i = 1, 2$, see Section 2.3, we note that

$$\begin{aligned} (\langle M_{nn}(v) \rangle, [v_{,n}])_E &= (\langle M_{nn}(v) \rangle, P_{l_1}[v_{,n}])_E, \\ (\langle T(v) \rangle, [v])_E &= (\langle T(v) \rangle, P_{l_2}[v])_E, \end{aligned}$$

since $\langle M_{nn}(v) \rangle$ and $\langle T(v) \rangle$ are polynomials of degree $p-2$ and $p-3$ on E, respectively. Using this observation, the Cauchy Schwarz inequality followed by the standard inequality $2ab < \epsilon a^2 + \epsilon^{-1} b^2$, for any positive ϵ, and finally the inverse inequalities (3.5), (3.6), and (3.7), we obtain

$$\begin{aligned} -\textstyle\sum_{E \in \mathcal{E} \setminus (\mathcal{E}_S \cup \mathcal{E}_F)} 2(\langle M_{nn}(v) \rangle, [v_{,n}])_E \quad \geq \quad & -\epsilon_1 C_1 |||v|||^2_{\mathcal{K}} \\ & -\epsilon_1^{-1} \|h^{-1} P_{l_1}[v_{,n}]\|^2_{\mathcal{E} \setminus (\mathcal{E}_S \cup \mathcal{E}_F)}, \end{aligned}$$

$$- \sum_{E \in \mathcal{E} \setminus \mathcal{E}_F} 2(\langle T(v) \rangle, [v])_E \geq -\epsilon_2 C_2 |||v|||^2_{\mathcal{K}} - \epsilon_2^{-1} \|h^{-2} P_{l_2}[v]\|^2_{\mathcal{E} \setminus \mathcal{E}_F},$$

$$-\sum_{E\in\mathcal{E}} 2(\langle M_{nt}(v)\rangle, n_{\partial E}[v])_{\partial E} \geq -\epsilon_3 C_3 |||v|||_{\mathcal{K}}^2 - \epsilon_3^{-1}\|h^{-1}[v]\|_{\partial\mathcal{E}}^2.$$

Given c, with $0 < c < 1$, we choose $\epsilon_i C_i = (1-c)/3$ and take $\beta_i \geq c + \epsilon_i^{-1}$, $i = 1, 2, 3$, and finally invoking the inverse estimates in Lemma 3.1 to bound $\|\langle M_{nn}(v)\rangle\|_{\mathcal{E}\setminus(\mathcal{E}_S\cup\mathcal{E}_F)}^2 + \|h\langle T(v)\rangle\|_{\mathcal{E}\setminus\mathcal{E}_F}^2 + \|\langle M_{nt}(v)\rangle\|_{\partial\mathcal{E}}^2$, we obtain the coercivity estimate (3.19). □

Theorem 3.1. *The dG approximation U, defined by (2.24), satisfies the following error estimate*

$$|||u-U||| \leq Ch^{p-1}\begin{cases}\|u\|_3 + h\|u\|_4 & p=2,\\ \|u\|_{p+1} & p\geq 3,\end{cases} \tag{3.23}$$

for sufficiently regular exact solution u of the plate equation. Furthermore, each element is in equilibrium, i.e., (2.27) holds, with the the discrete moments and transversal forces, defined in (2.19)–(2.21), and the following error estimates hold

$$\|M_{nn}(u) - M_{nn}(U)\|_{\mathcal{E}} \leq Ch^{p-1}\|u\|_{p+1}, \tag{3.24}$$

$$\|T(u) - T(U)\|_{\mathcal{E}} \leq Ch^{p-2}\|u\|_{p+1}, \tag{3.25}$$

$$\|M_{nt}(u) - M_{nt}(U)\|_{\partial\mathcal{E}} \leq Ch^{p-1}\|u\|_{p+1}, \tag{3.26}$$

where for $p = 2$ we replace $\|u\|_3$ by $\|u\|_3 + h\|u\|_4$. The constant C is independent of h but may in general depend on β.

Proof. To prove the energy norm error estimate (3.29) we first use the triangle inequality

$$|||u-U||| \leq |||u-\pi u||| + |||\pi u - U|||,$$

where $\pi u \in \mathcal{W}$ is an interpolant of u. Here the first term can be estimated immediately using the interpolation error estimate (3.13). Next, using coercivity, consistency, and finally the continuity (see Lemma 3.4) of the bilinear form we get

$$\begin{aligned} c|||\pi u - U|||^2 &\leq a(\pi u - U, \pi u - U)\\ &= a(\pi u - u, \pi u - U)\\ &\leq C|||\pi u - u|||\,|||\pi u - U|||. \end{aligned}$$

Dividing by $|||\pi u - U|||$ and again employing the interpolation error estimate completes the proof.

The estimates of the normal moment, the discrete transversal force, and the twisting moment follows from the energy norm error estimate. For instance,

$$\begin{aligned} \|M_{nn}(u) - M_{nn}(U)\|_{\mathcal{E}} &\leq \|M_{nn}(u) - \langle M_{nn}(U)\rangle\|_{\mathcal{E}} + \|h^{-2}P_{l_1}[U]\|_{\mathcal{E}}\\ &\leq 2|||u-U|||, \end{aligned}$$

and similarly for the transversal force and twisting moment. □

We now turn to an estimate of the L^2 norm of the error. We assume that there is $\phi \in H^4$ such that

$$a(v, \phi) = (v, \psi), \tag{3.27}$$

for all $v, \psi \in \mathcal{V}$, and that the stability estimate

$$\|\phi\|_4 \leq C\|\psi\|, \tag{3.28}$$

holds, see Blum and Rannacher [5].

Theorem 3.2. *If the stability estimate (3.28) holds, then U satisfies*

$$\|u - U\| \leq C \begin{cases} h^2\Big(\|u\|_3 + h\|u\|_4\Big) & p = 2, \\ h^{p+1}\|u\|_{p+1} & p \geq 3, \end{cases} \tag{3.29}$$

for sufficiently regular u. The constant C is independent of h but may in general depend on β.

Proof. Setting $v = \psi = u - U$, in the dual problem (3.27) and using Galerkin orthogonality (2.24) to subtract an interpolant $\pi\phi$ of ϕ we obtain

$$\begin{aligned} \|u - U\|^2 &= a(u - U, \phi) \\ &= a(u - U, \phi - \pi\phi) \\ &\leq C|||u - U|||\;|||\phi - \pi\phi||, \end{aligned}$$

where we used the (3.18) in the last step. Next using the energy norm error estimate (3.29) in Theorem 3.1 and the interpolation error estimate (3.13) we have

$$\|u - U\|^2 \leq C \begin{cases} h^2\Big(\|u\|_3 + h\|u\|_4\Big)\Big(\|\phi\|_3 + h\|\phi\|_4\Big) & p = 2, \\ h^{p+1}\|u\|_{p+1}\|\phi\|_4 & p \geq 3, \end{cases}$$

which together with the stability estimate (3.28) concludes the proof. □

3.3. Relation to classical methods.

3.3.1. *The nonconforming Morley element.* For a quadratic approximation, $p = 2$, we have that $M_{nn}(U)$ is a constant and $T(U)$ is zero on each edge, so we may choose $P_{l_1} = P_0$, the projection on constants, and $P_{l_2} = P_{-1}$, i.e., P_{l_2} is the zero projection. For large values of the parameters β_1 we thus enforce the following continuity condition on each edge

$$([u_{,n}], v)_E = 0 \quad \text{for all } v \in \mathcal{P}_0(E), \tag{3.30}$$

and for large values of β_3 nodal continuity is enforced. See, Larson and Niklasson [11], for an analysis of the behavior of the dG method as the parameter $\beta \to \infty$, in the case of the Poisson equation.

These continuity requirements define the classical nonconforming quadratic Morley element [12, 13].

Note that with these choices of P_{l_1} and P_{l_2} we get a bound in Theorem 3.1 which is independent of β_i, and thus locking does not occur when β_i tends to infinity. However, for a quadratic approximation, $p = 2$, and all other choices of l_1 and l_2 the method will lock for large β.

3.3.2. *The C^1 Argyris element.* The fifth order Argyris triangle is a classical C^1 element, see [1]. Using this element together with $P_{l_1} = P_{l_2} = I$, the identity projection, yields error estimates which are not dependent on β. Letting $\beta_i \to \infty$ we obtain the C^1 method based on the Argyris element.

4. Numerical examples

4.1. **Convergence.** We consider a square plate with side length $L = 1$, thickness $t = 1/10$, Poisson's ration $\nu = 1/2$ and Young's modulus $E = 1$. The plate is loaded with a linearly varying surface load $f = x_1$, and the boundary conditions are $u = u_{,n} = 0$ on $x_1 = 0$ and $x_1 = 1$, and $M_{nn} = T = 0$ at $x_2 = 0$ and $x_2 = 1$. The solution to this problem can be found by direct integration:

$$u(x_1, x_2) = \frac{7\,x_1}{360\,D} - \frac{x_1^3}{36\,D} + \frac{x_1^5}{120\,D}.$$

To show the convergence pattern, we have tested using second degree polynomial and third degree polynomial approximations. The choice of stabilization parameters was $\beta_i = 5$. Since the solution is smooth we expect optimal convergence rates. This was also found, as can be seen in Figure 1, indicating second order convergence in $L_2(\Omega)$–norm for the second degree polynomials and fourth order for the third degree polynomials.

4.2. **Locking.** To show that locking can occur if the stabilization parameters are chosen too large, we consider the same problem as in Section 4.1, with different boundary conditions, $u = u_{,n} = 0$ on $\partial\Omega$. The solution was computed on a fixed grid, shown in Figure 2.

We make a comparison with the Morley approximation as follows. We choose $\beta_2 = 0$ and study the behavior of the solution for increasing $\beta_1 = \beta_3 =: \beta$. We compare one point Gaussian integration with two point Gaussian integration. One point integration directly corresponds to the Morley element as $\beta \to \infty$, since this will enforce normal derivative continuity at the midpoints of the edges plus nodal continuity. Thus the one point rule will not lock. The two point integration scheme, however, may tend to overconstrain the problem. This is clearly seen in Figure 3, where locking is obtained in that the level of the solution decreases with increasing β for the two point rule (dashed line), whereas no locking occurs for the one point rule (solid line).

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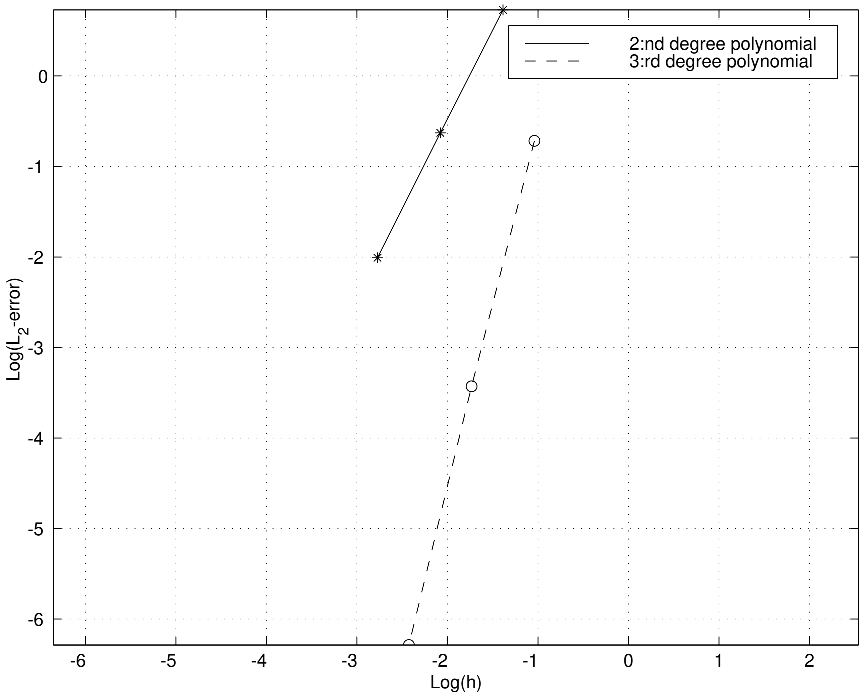

FIGURE 1. Convergence rates in the L_2–norm for second and third degree polynomial approximation

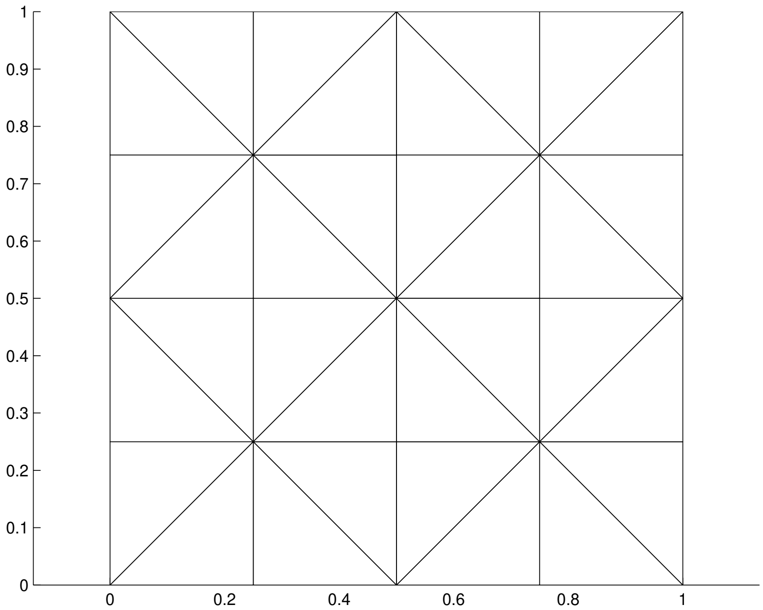

FIGURE 2. Mesh used to illustrate locking.

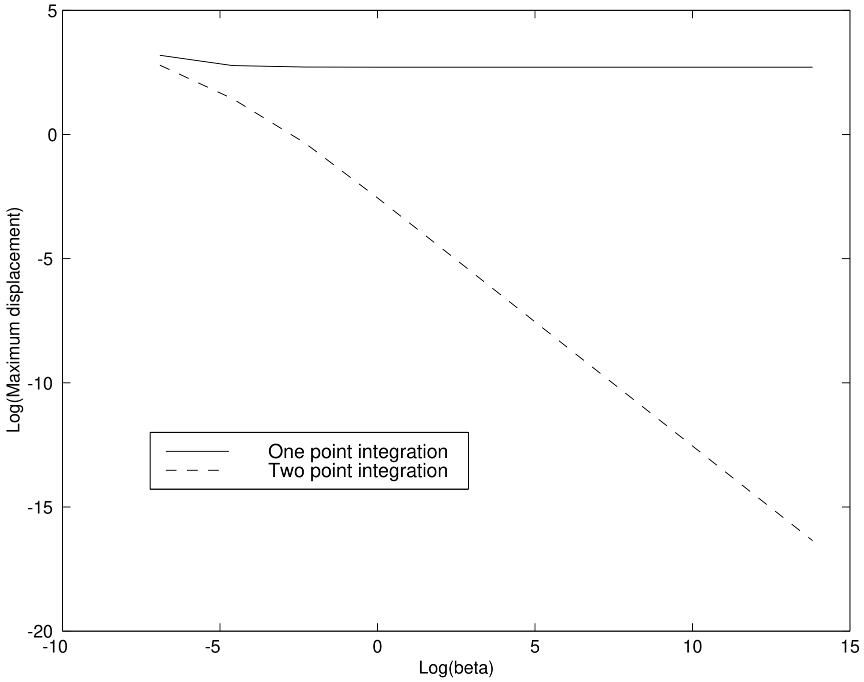

FIGURE 3. Locking for the two point integration rule but not for the one point.

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