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Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg
Sweden
Telephone: +46 (0)31 772 1000
Fax: +46 (0)31 772 3595
www.phi.chalmers.se

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Energy Norm A Posteriori Error Estimation for Discontinuous Galerkin Methods

Roland Becker* Peter Hansbo† Mats G. Larson‡

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Abstract

In this note we present a residual-based a posteriori error estimate of a natural mesh dependent energy norm of the error in a family of discontinuous Galerkin approximations of elliptic problems. The theory is developed for an elliptic model problem in two and three spatial dimensions and general nonconvex polygonal domains are allowed. We also present some illustrating numerical examples.

1 Introduction

Discontinuous Galerkin (dG) methods for elliptic problems have recently received renewed interest, see [4] for an overview. One of the advantages is the flexible construction of approximation spaces, for instance allowing non-matching grids and different order of polynomials on bordering elements without continuity enforcement. This property make the dG method attractive for using together with an adaptive algorithm. Adaptive algorithms are in general based on a posteriori error estimates providing information on where local refinement is necessary.

In this paper we derive an a posteriori error estimate, of residual type, of a natural mesh dependent energy norm. The estimate is of optimal order and is valid for a general family of dG methods including the classical symmetric Nitsche method [11], the recent nonsymmetric method proposed by Oden, Babuška, and Baumann [12], and stabilized versions thereof.

Other work on a posteriori estimates for discontinuous Galerkin methods include Becker, Hansbo, and Stenberg [1] where a weighted residual estimator of the L^2-norm of the error is presented and Rivière and Wheeler [13], where a residual estimator is derived for L^2-norm of the error and an implicit a posteriori error estimate of the energy norm based on local

*Institut für Angewandte Mathematik, Universität Heidelberg, Neuenheimer Feld 294, D-69120 Heidelberg, Germany

†Department of Solid Mechanics, Chalmers University of Technology, Göteborg, Sweden

‡Department of Mathematics, Chalmers University of Technology, Göteborg, Sweden

problems is evaluated numerically. It turns out that deriving a posteriori estimates of the L^2-norm of the error using duality arguments is indeed much simpler than energy norm a posteriori error estimates. This is a consequence of the fact that the solution of the dual problem is more regular than the discontinuous approximation.

Key to our proof of the energy norm a posteriori error estimate is a Helmholtz decomposition of the gradient of the error. This technique is used to prove a posteriori error estimates for nonconforming finite element methods by Dari, Duran, Padra, and Vampa [5] and Carstensen, Bartels, and Jansche [3]. Our derivation of the a posteriori error estimate contains two novel details. First, the usual Galerkin orthogonality principle is replaced by the use of the fact that the dG method provides an explicit elementwise conservative normal flux, and second our technique to handle the case of nonconvex polyhedra in three dimensions which is related to the argument for nonconforming elements in Carstensen et al [3]. The resulting a posteriori error estimate is also somewhat simpler than corresponding results for nonconforming elements.

The paper is organized as follows: in Section 2 we introduce a model problem and the family of dG methods; in Section 3 we derive our a posteriori estimate; and finally in Section 4 we present some numerical experiments.

2 The model problem and the dG method

2.1 Model problem

We consider the following boundary value problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla \cdot \sigma(u) &= f \quad \text{in } \Omega, \\ u &= g_D \quad \text{on } \Gamma_D, \\ \sigma_n(u) := n \cdot \sigma(u) &= g_N \quad \text{on } \Gamma_N, \end{aligned} \tag{2.1}$$

where Ω denotes a bounded polygonal domain in $\mathbb{R}^d$, $d = 2$ or 3, with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, and $\sigma(u)$ is defined by

$$\sigma(u) = \nabla u. \tag{2.2}$$

If $\Gamma_D \neq \emptyset$, (2.1) has a unique solution $u \in H^1$ for each $f \in H^{-1}$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in H^{-1/2}(\Gamma_N)$, and if $\Gamma_D = \emptyset$, the solution exists and is unique up to a constant, i.e., $u \in H^1/\mathbb{R}$ if $f \in H^{-1}$, $g_N \in H^{1/2}(\Gamma)$, provided the compatibility condition

$$\int_\Omega f + \int_\Gamma g_N = 0, \tag{2.3}$$

is satisfied.

2.2 Discontinuous spaces

To define the dG method we introduce a partition $\mathcal{K} = \{K\}$ of Ω, satisfying the minimal angle condition, see [2]. Further, we let the mesh function $h : \Omega \to (0, \infty)$ be defined by $h|_K = h_K = \text{diam}(K)$. We let $\mathcal{DP}$ be the space of discontinuous piecewise polynomials defined on $\mathcal{K}$:

$$\mathcal{DP} = \bigoplus_{K \in \mathcal{K}} \mathcal{P}_{p_K}(K), \tag{2.4}$$

where $\mathcal{P}_{p_K}(K)$ is a space of polynomials of degree p_K defined on the element K. Note that the order of polynomials is allowed to vary from element to element.

2.3 The dG method

The dG method for (2.1) is defined by: find $U \in \mathcal{DP}$ such that

$$a(U, v) = l(v) \quad \text{for all } v \in \mathcal{DP}, \tag{2.5}$$

where $a(\cdot, \cdot)$ and $l(\cdot)$ are sums of elementwise defined forms

$$a(v, w) := \sum_{K \in \mathcal{K}} a_K(v, w), \qquad l(v) := \sum_{K \in \mathcal{K}} l_K(v), \tag{2.6}$$

given by

$$\begin{aligned} a_K(v, w) := &(\sigma(v), \nabla w)_K - (\langle \sigma_n(v) \rangle, w)_{\partial K \setminus \Gamma_N} \\ &+ \alpha([v], \sigma_n(w))_{\partial K \setminus \Gamma_N}/2 + \beta(h^{-1}[v], w)_{\partial K \setminus \Gamma_N}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} l_K(w) := &(f, w)_K + (g_N, w)_{\partial K \cap \Gamma_N} \\ &+ \alpha(g_D, \sigma_n(w))_{\partial K \cap \Gamma_D} + \beta(g_D, h^{-1} w)_{\partial K \cap \Gamma_D}, \end{aligned} \tag{2.8}$$

where α and β are real constants. We also employed the notation

$$\langle v \rangle := \begin{cases} (v^+ + v^-)/2 & \text{on } \partial K \setminus \Gamma, \\ v^+ & \text{on } \partial K \cap \Gamma_D, \end{cases} \tag{2.9}$$

and

$$[v] := \begin{cases} v^+ - v^- & \text{on } \partial K \setminus \Gamma, \\ v^+ & \text{on } \partial K \cap \Gamma_D, \end{cases} \tag{2.10}$$

where $v^\pm(x) = \lim_{s \to 0^+} v(x \mp s n_K)$. Further on each edge $E = K^+ \cap K^-$, the mesh parameter h is defined by

$$h := \frac{m(K^+) + m(K^-)}{3m(E)}, \tag{2.11}$$

where $m(\cdot)$ denotes the appropriate Lebesgue measure. Simpler choices such as $h = \langle h \rangle$ may also be used in the case of locally quasiuniform meshes.

2.4 The mesh dependent energy norm

We introduce the following mesh dependent energy norm

$$|||v|||^2 = |||v|||_{\mathcal{K}}^2 + |||v|||_{\partial\mathcal{K}}^2, \tag{2.12}$$

where

$$|||v|||_{\mathcal{K}}^2 = \sum_{K\in\mathcal{K}} (\sigma(v), \nabla v)_K, \tag{2.13}$$

$$|||v|||_{\partial\mathcal{K}}^2 = \sum_{K\in\mathcal{K}} (h^{-1}[v], [v])_{\partial K\setminus\Gamma}/2 + (h^{-1}v, v)_{\partial K\cap\Gamma_D}. \tag{2.14}$$

2.5 Stability and a priori error estimates

We shall assume that α and β are chosen such that the inf-sup condition

$$m \leq \inf_{v\in\mathcal{DP}} \sup_{w\in\mathcal{DP}} \frac{a(v,w)}{|||v|||\,|||w|||}, \tag{2.15}$$

holds with constant m independent of h. For instance, for the classical Nitsche method [11], $\alpha = -1$ and (2.15) is satisfied for sufficiently large β. Taking $\alpha = 1$ we obtain the nonsymmetric method suggested by Oden, Babuška and Baumann [12] and (2.15) is satisfied for $\beta > 0$. In fact, for quadratic and higher order polynomials β can be set to zero in the nonsymmetric case, see [8] and [9].

If (2.15) holds one can prove the a priori error estimate

$$|||u - U||| \leq ch^{\mu-1}|u|_l, \tag{2.16}$$

where $\mu = \min(l, p+1)$ and $u \in H^l$.

2.6 Elementwise conservation property

Introducing the discrete flux

$$\Sigma_n(U) := \begin{cases} \langle\sigma_n(U)\rangle - \beta h^{-1}[U] & \text{on } \partial K \setminus \Gamma, \\ \sigma_n(U) - \beta h^{-1}(U - g_D) & \text{on } \partial K \cap \Gamma_D, \\ g_N & \text{on } \partial K \cap \Gamma_N, \end{cases} \tag{2.17}$$

we obtain the discrete elementwise conservation law

$$\int_K f + \int_{\partial K} \Sigma_n(U) = 0, \tag{2.18}$$

for all $K \in \mathcal{K}$. Thus the directly accessible flux $\Sigma_n(U)$ is elementwise conservative. This property will play an important role in our later developments.

3 A posteriori error estimate of the energy norm

3.1 Error representation

We begin with a lemma giving a Helmoltz decomposition of the elementwise defined flux $\sigma(e)$. We treat the two- and three-dimensional cases simultaneously and employ the notation

$$\operatorname{curl} v = \begin{cases} (\partial_2 v, -\partial_1 v) & d = 2, \\ \nabla \times v & d = 3, \end{cases} \tag{3.1}$$

for $v \in [H^1(\Omega)]^k$ with $k = 1$ if $d = 2$ and $k = 3$ if $d = 3$.

Lemma 3.1 *There exists $\phi \in H^1(\Omega)$ and $\chi \in [H^1(\Omega)]^k$ such that*

$$\sigma(e) = \sigma(\phi) + \operatorname{curl} \chi, \tag{3.2}$$

with

$$\phi = 0 \text{ on } \Gamma_D \text{ and } n \cdot \operatorname{curl} \chi = 0 \text{ on } \Gamma_N, \tag{3.3}$$

and the stability estimate

$$\|\sigma(\phi)\| + \|\operatorname{curl} \chi\| \leq c|||e|||_{\mathcal{K}}. \tag{3.4}$$

holds

PROOF. Let $\phi \in \mathcal{V}_0 = \{H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ be defined by

$$(\sigma(\phi), \nabla v) = (\sigma(e), \nabla v), \tag{3.5}$$

for all $v \in \mathcal{V}_0$. Then $\nabla \cdot (\sigma(\phi) - \sigma(e)) = 0$ and $[n \cdot (\sigma(\phi) - \sigma(e))] = 0$ at each edge. Further we have $(\sigma_n(\phi), 1)_{\Gamma_N} = 0$ and thus $\sigma(\phi) - \sigma(e) = \operatorname{curl} \chi$, see Girault and Raviart [7]. $\square$

Note that since $|||e|||_{\partial\mathcal{K}} = |||U|||_{\partial\mathcal{K}}$ is directly computable we have

$$|||e|||^2 = |||e|||_{\mathcal{K}}^2 + |||U|||_{\partial\mathcal{K}}^2, \tag{3.6}$$

and thus we need to estimate $|||e|||_{\mathcal{K}}$ to obtain an a posteriori estimate of $|||e|||$.

Using Lemma 3.1 we get

$$|||e|||_{\mathcal{K}}^2 = \sum_{K \in \mathcal{K}} (\nabla e, \sigma(\phi))_K + (\nabla e, \operatorname{curl} \chi)_K. \tag{3.7}$$

For the first term on the right hand side of (3.7) we begin by observing that $(\nabla e, \sigma(\phi)) = (\sigma(e), \nabla \phi)$, and subtracting a piecewise constant projection $\pi_0 \phi$ of ϕ, which is zero on all elements K with a face on Γ_D. Then we integrate by parts

$$\begin{aligned} \sum_{K \in \mathcal{K}} (\sigma(e), \nabla \phi)_K &= \sum_{K \in \mathcal{K}} (\sigma(e), \nabla(\phi - \pi_0 \phi))_K & (3.8) \\ &= \sum_{K \in \mathcal{K}} (f + \nabla \cdot \sigma(U), \phi - \pi_0 \phi)_K & (3.9) \\ &\qquad + (\sigma_n(u) - \sigma_n(U), \phi - \pi_0 \phi)_{\partial K \setminus \Gamma_D} \\ &= \sum_{K \in \mathcal{K}} (f + \nabla \cdot \sigma(U), \phi - \pi_0 \phi)_K & (3.10) \\ &\qquad + (\Sigma_n(U) - \sigma_n(U), \phi - \pi_0 \phi)_{\partial K \setminus \Gamma_D}. \end{aligned}$$

In the last equality we used the fact that $\Sigma_n(U)$ is continuous accross edges so that $\sum_{K \in \mathcal{K}} (\sigma_n(u), \phi)_{\partial K \setminus \Gamma_D} = \sum_{K \in \mathcal{K}} (\Sigma_n(u), \phi)_{\partial K \setminus \Gamma_D}$ and conservative so that $(\sigma_n(u), \pi_0 \phi)_{\partial K \setminus \Gamma_D} = (\Sigma_n(U), \pi_0 \phi)_{\partial K \setminus \Gamma_D}$ for all elements $K \in \mathcal{K}$ without a face on Γ_D.

Next for the second term on the right hand side of (3.7) we integrate by parts

$$\begin{aligned} \sum_{K \in \mathcal{K}} (\nabla e, \operatorname{curl} \chi)_K &= \sum_{K \in \mathcal{K}} (u - U, n \cdot \operatorname{curl} \chi)_{\partial K \setminus \Gamma_N} & (3.11) \\ &= \sum_{K \in \mathcal{K}} (v - U, n \cdot \operatorname{curl} \chi)_{\partial K \setminus \Gamma_N}. & (3.12) \end{aligned}$$

Here we replaced u by an arbitrary function $v \in \mathcal{V}_{g_D}$, where

$$\mathcal{V}_{g_D} = \{v \in H^1 : v = g_D \text{ on } \Gamma_D\}, \tag{3.13}$$

in equality (3.12). The replacement is allowed since the normal component of $\operatorname{curl} \chi$ is continuous and $v = u$ on Γ_D.

Together (3.7), (3.10), and (3.12), give the representation formula

$$\begin{aligned} |||e|||_{\mathcal{K}}^2 = \sum_{K \in \mathcal{K}} &(f + \nabla \cdot \sigma(U), \phi - \pi_0 \phi)_K & (3.14) \\ &+ (\Sigma_n(U) - \sigma_n(U), \phi - \pi_0 \phi)_{\partial K \setminus \Gamma_D} \\ &+ (v - U, n \cdot \operatorname{curl} \chi)_{\partial K \setminus \Gamma_N}, \end{aligned}$$

for all $v \in \mathcal{V}_{g_D}$.

3.2 An a posteriori error estimate

We are now ready to state our a posteriori error estimate.

Theorem 3.1 *The error $e = u - U$ satisfies*

$$|||e|||^2 \leq c\left(\sum_{K\in\mathcal{K}} \rho_K^2\right), \tag{3.15}$$

with constant c independent of h and element indicator ρ_K defined by

$$\begin{aligned}\rho_K^2 = h_K^2\|f + \nabla\cdot\sigma(U)_K\|_K^2 \\ + h_K\|\Sigma_n(U) - \sigma_n(U)\|^2_{\partial K\setminus\Gamma_D} + h_K^{-1}\|[U]\|^2_{\partial K\setminus\Gamma_N},\end{aligned} \tag{3.16}$$

where the discrete normal flux $\Sigma_n(U)$ is defined by

$$\Sigma_n(U) = \begin{cases}\langle\sigma_n(U)\rangle - \beta h^{-1}[U] & \text{on } \partial K\setminus\Gamma,\\ g_N & \text{on } \partial K\cap\Gamma_N.\end{cases} \tag{3.17}$$

We shall need the following lemma in the proof of Theorem 3.1.

Lemma 3.2 *It holds*

$$\inf_{v\in\mathcal{V}_{g_D}} \sum_{K\in\mathcal{K}} \|v - U\|^2_{1/2,\partial K\setminus\Gamma_N} \leq c\left(\sum_{K\in\mathcal{K}} h_K^{-1}\|[U]\|^2_{\partial K\setminus\Gamma_N}\right), \tag{3.18}$$

with constant c independent of h and $\mathcal{V}_{g_D}$ defined in (3.13).

PROOF OF THEOREM 3.1. The proof consists of estimates of the three terms on the right hand side of (3.14). We begin by observing that using the Cauchy-Schwarz inequality followed by standard estimates we obtain

$$\|\phi - \pi_0\phi\| \leq ch_K\|\nabla\phi\| \leq ch_K\|\sigma(\phi)\|. \tag{3.19}$$

For the first term we get, using the Cauchy-Schwarz inequality and (3.19),

$$|(f + \nabla\cdot\sigma(U), \phi - \pi_0\phi)_K| \leq \|f + \nabla\cdot\sigma(U)\|_K\|\phi - \pi_0\phi\|_K \tag{3.20}$$

$$\leq ch_K\|f + \nabla\cdot\sigma(U)\|_K\|\sigma(\phi)\|_K. \tag{3.21}$$

Next to estimate the second term term we use the Cauchy-Schwarz inequality, the trace inequality $\|v\|^2_{\partial K} \leq c\|v\|(h_K^{-1}\|v\| + \|\nabla v\|)$, and (3.19) to get

$$\begin{aligned}|(\Sigma_n(U) - \sigma_n(U), \phi - \pi_0\phi)_{\partial K\setminus\Gamma_D}| &\leq \|\Sigma_n(U) - \sigma_n(U)\|_{\partial K\setminus\Gamma_D}\|\phi - \pi_0\phi\|_{\partial K\setminus\Gamma_D}\\ &\leq ch_K^{1/2}\|\Sigma_n(U) - \sigma_n(U)\|_{\partial K\setminus\Gamma_D}\|\sigma(\phi)\|_K.\end{aligned} \tag{3.22}$$

For the last term we have

$$\begin{aligned}|(v - U, n\cdot\operatorname{curl}\chi)_{\partial K\setminus\Gamma_N}| &\leq \|v - U\|_{H^{1/2}(\partial K\setminus\Gamma_N)}\|n\cdot\operatorname{curl}\chi\|_{H^{-1/2}(\partial K\setminus\Gamma_N)}\\ &\leq c\|v - U\|_{H^{1/2}(\partial K\setminus\Gamma_N)}\|\operatorname{curl}\chi\|_K.\end{aligned} \tag{3.23}$$

In (3.23) we employed the trace inequality

$$\|n \cdot \operatorname{curl} \chi\|_{H^{-1/2}(\partial K)} \leq c\|\operatorname{curl} \chi\|_K, \tag{3.24}$$

which follows from

$$\|n \cdot w\|_{H^{-1/2}(\partial K)} \leq c\Big(\|w\|_K + h_K\|\nabla \cdot w\|_K\Big), \tag{3.25}$$

see [7], with $w = \operatorname{curl} \chi$, together with the fact that $\nabla \cdot \operatorname{curl} \chi = 0$.

Summing over all the elements, using the stability estimate (3.4) and dividing by $|||e|||_{\mathcal{K}}$ we obtain

$$\begin{aligned} |||e|||^2 \leq \sum_{K\in\mathcal{K}} & h_K^2\|f + \nabla \cdot \sigma(U)_K\|_K^2 \\ & + h_K\|\Sigma_n(U) - \sigma_n(U)\|^2_{\partial K\setminus\Gamma_D} + \|v - U\|^2_{1/2,\partial K\setminus\Gamma_N}, \end{aligned} \tag{3.26}$$

for all $v \in \mathcal{V}_{g_D}$. By Lemma 3.2 we may replace $\inf_{v\in\mathcal{V}_{g_D}} \sum_{K\in\mathcal{K}} \|v - U\|^2_{H^{1/2}(\partial K\setminus\Gamma_N)}$ by $\sum_{K\in\mathcal{K}} h_K^{-1}\|[U]\|^2_{\partial K\setminus\Gamma_N}$, and thus the proof is complete.

PROOF OF LEMMA 3. Let $\mathcal{X} = \{X\}$ be the set of nodes, $\{\varphi_X\}$ the set of associated piecewise linear (or bilinear) basis functions, $\omega_X = \operatorname{supp}(\varphi_X)$, $\mathcal{CP}_X = C(\omega_X) \cap \{v : v = w|_{\omega_X}, w \in \mathcal{DP}, w = g_D \text{ on } \Gamma_D\}$, and finally $\mathcal{CP} = \bigoplus_{K\in\mathcal{K}} \varphi_X\mathcal{CP}_X \subset \mathcal{V}_{g_D}$.

We clearly have

$$\inf_{v\in\mathcal{V}_{g_D}} \sum_{K\in\mathcal{K}} \|v - U\|^2_{H^{1/2}(\partial K\setminus\Gamma_N)} \leq \inf_{v\in\mathcal{CP}} \sum_{K\in\mathcal{K}} \|v - U\|^2_{H^{1/2}(\partial K\setminus\Gamma_N)}. \tag{3.27}$$

Using an inverse inequality we have

$$\|v - U\|_{H^{1/2}(\partial K\setminus\Gamma_N)} \leq ch_K^{-1/2}\|v - U\|_{\partial K\setminus\Gamma_N}, \tag{3.28}$$

for all $v \in \mathcal{CP}$ and $K \in \mathcal{K}$. Further, for each $v = \sum_{X\in\mathcal{X}} \varphi_X v_X \in \mathcal{CP}$ we have

$$\|v - U\|^2_{\partial K\setminus\Gamma_N} = \sum_{X\in\mathcal{X}} (v - U, \varphi_X(v_X - U))_{\partial K\setminus\Gamma_N} \tag{3.29}$$

$$\leq \sum_{X\in\mathcal{X}} \|\varphi_X^{1/2}(v - U)\|_{\partial K\setminus\Gamma_N}\|\varphi_X^{1/2}(v_X - U)\|_{\partial K\setminus\Gamma_N} \tag{3.30}$$

$$\leq \|v - U\|_{\partial K\setminus\Gamma_N} \left(\sum_{X\in\mathcal{X}} \|\varphi_X^{1/2}(v_X - U)\|^2_{\partial K\setminus\Gamma_N}\right)^{1/2}, \tag{3.31}$$

where we used the fact that $\sum_{X\in\mathcal{X}} \varphi_X = 1$ and the Cauchy-Schwarz inequality. We can conclude that

$$\sum_{K\in\mathcal{K}} h_K^{-1}\|v - U\|^2_{\partial K\setminus\Gamma_N} \leq \sum_{X\in\mathcal{X}} \sum_{K\in\mathcal{K}} h_K^{-1}\|\varphi_X^{1/2}(v_X - U)\|^2_{\partial K\setminus\Gamma_N}, \tag{3.32}$$

for all $v \in \mathcal{CP}$.

Next we note that there is a constant c independent of h such that

$$\inf_{v_X \in \mathcal{CP}_X} \sum_{K \in \mathcal{K}} h_K^{-1} \|\varphi_X^{1/2}(v_X - U)\|^2_{\partial K \setminus \Gamma_N} \leq \sum_{K \in \mathcal{K}} h_K^{-1} \|\varphi_X^{1/2}[U]\|^2_{\partial K \setminus \Gamma_N}. \tag{3.33}$$

Inequality (3.33) follows from the fact that if the right hand side is zero then U is continuous on ω_X and we may take $v_X = U|_{\omega_X}$. Then the left hand side is also zero. Now (3.32) imediately follows by finite dimensionality and scaling.

Finally, (3.18) follows from (3.27), (3.32) and (3.33) together with the identity

$$\sum_{X \in \mathcal{X}} \sum_{K \in \mathcal{K}} h_K^{-1} \|\varphi_X^{1/2}[U]\|^2_{\partial K \setminus \Gamma_N} = \sum_{K \in \mathcal{K}} h_K^{-1} \|[U]\|^2_{\partial K \setminus \Gamma_N}, \tag{3.34}$$

where, again, we used the fact that $\sum_{X \in \mathcal{X}} \varphi_X = 1$.

4 Examples

In all the computational examples, we have used linear discontinuous elements on triangular, geometrically conforming meshes. We have chosen the constants as $c = 1$ (in (3.15)) and $\beta = 5$ in all computations, and we have discarded the internal residuals, so that the effectivity indices are based only on the jump terms.

4.1 Sinusoidal hill

We consider the problem (2.1) with data $f = 2\,\pi^2 \sin(\pi\, x) \sin(\pi\, y)$ in the domain $\Omega = (0,1) \times (0,1)$, with $\Gamma_N = \emptyset$ and $g_D = 0$. This problem has the exact solution $u = \sin(\pi\, x) \sin(\pi\, y)$.

In Figure 4.2, we show the last mesh used in the computations using the unsymmetric bilinear form, in Figure 2 we show an elevation of the corresponding solution, and in Figure 3 we show the effectivity index on the successively refined meshes using the symmetric and unsymmetric bilinear form corresponding to $\alpha = \pm 1$. We note that the error is overestimated by around a factor of 3 with the choices of constants made. We emphasize that we do not attempt to reach effectivity indices of around one, but rather to show that the effectivity index does not vary too much with respect to a given mesh. We refine the mesh using an adaptive algorithm based on refining a fixed fraction (30%) of those element with the largest error contribution.

4.2 Peak function

For our next example, we choose the "peak function" used for instance by Rivière and Wheeler [13], given by the exact solution

$$u = \frac{e^{10\,x^2 + 10\,y}\,(1-x)^2\,x^2\,(1-y)^2\,y^2}{2000}, \tag{4.1}$$

on the domain $(0,1) \times (0,1)$. The parameters and method for refining the mesh are the same as in the previous example. In Figure 4, we give the last mesh in a sequence of refined meshes using the symmetric bilinear form, and in Figure 5 we give the isolines of the corresponding solution. Finally, in Figure 6, we give the variation of the effectivity index on the refined meshes for both the symmetric and unsymmetric forms.

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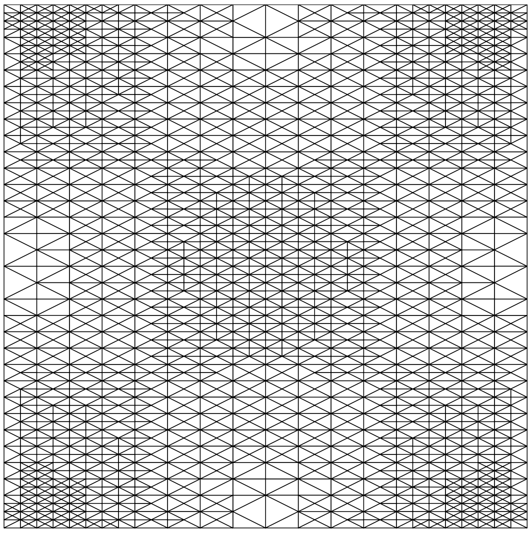

Figure 1: Final adapted mesh for the sine function.

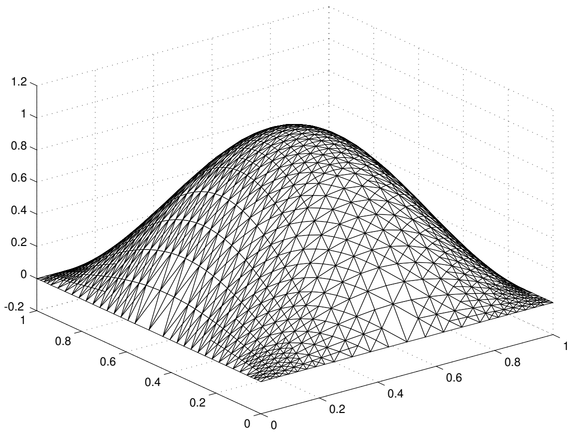

Figure 2: Elevation of the discrete solution.

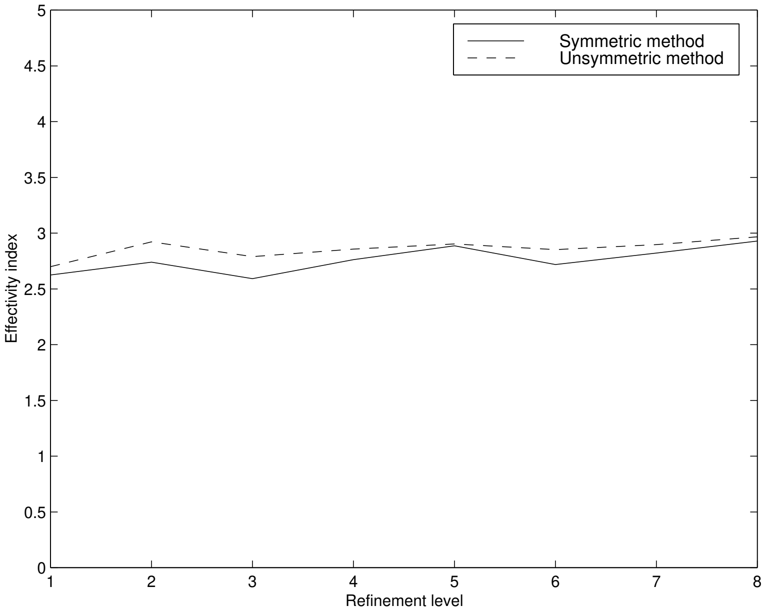

Figure 3: Effectivity index as a function of the refinement level.

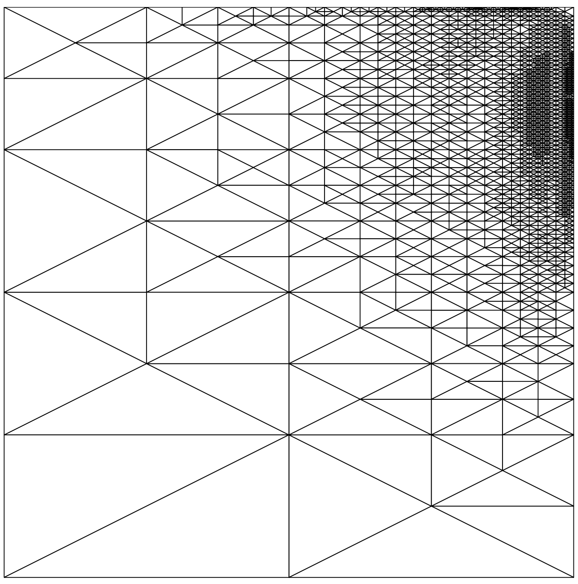

Figure 4: Final adapted mesh for the peak function.

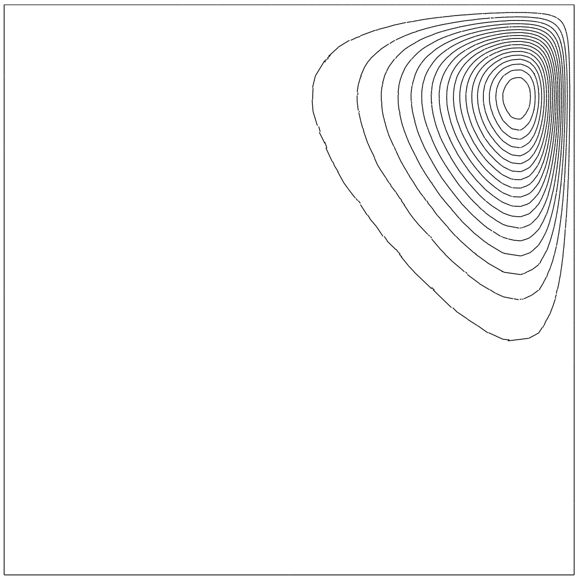

Figure 5: Isolines of the discrete solution.

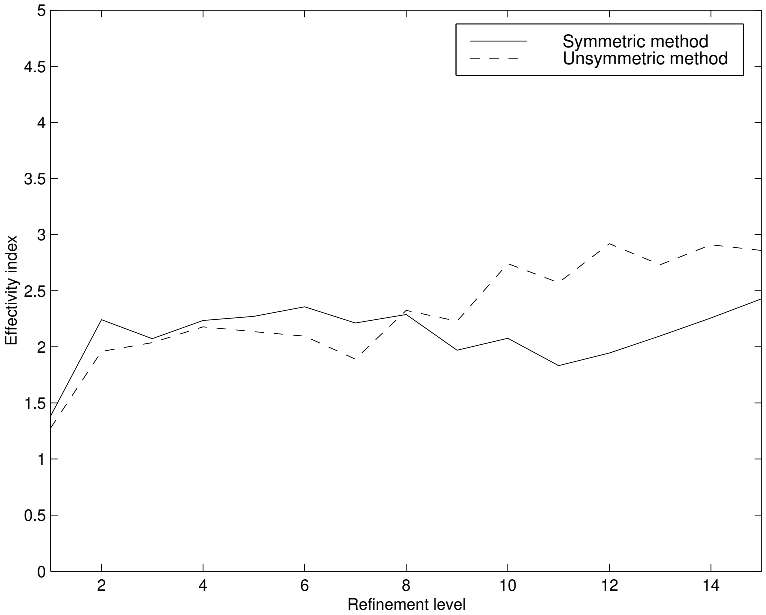

Figure 6: Effectivity index as a function of the refinement level.

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