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Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg
Sweden
Telephone: +46 (0)31 772 1000
Fax: +46 (0)31 772 3595
www.phi.chalmers.se

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Analysis of a Family of Discontinuous Galerkin Methods for Elliptic Problems: the One Dimensional Case *

Mats G. Larson † A. Jonas Niklasson ‡

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Abstract

In this paper we analyze a family of discontinuous Galerkin methods, parametrized by two real parameters, for elliptic problems in one dimension. Our main results are: (1) a complete inf-sup stability analysis characterizing the parameter values yielding a stable scheme and energy norm error estimates as a direct consequence thereof, (2) an analysis of the error in L^2 where the standard duality argument only works for special parameter values yielding a symmetric bilinear form and different orders of convergence are obtained for odd and even order polynomials in the nonsymmetric case. The analysis is consistent with numerical results and similar behaviour is observed in two dimensions.

1 Introduction

Discontinuous Galerkin methods have recently obtained renewed interest, motivated by several appealing properties as well as the successful application to hyperbolic problems. See the recent conference proceedings [5] for an overview of recent developments in this area.

This paper grew out of an effort to analytically understand the properties of discontinuous Galerkin methods for elliptic problems. We present an analysis in one spatial dimension which we believe give new insights guiding the analysis in two dimensions, which we present in a second paper. Throughout the paper we make an effort to present illustrating numerical results verifying our analytical results.

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†Corresponding author, Department of Mathematics, Chalmers University of Technology, Göteborg, SE-412 96, Sweden, mgl@math.chalmers.se

‡Department of Applied Mechanics, Chalmers University of Technology, Göteborg, SE-412 96, Sweden, jonas.niklasson@me.chalmers.se

We shall study a family of discontinuous Galerkin methods for second order elliptic problems. The family contains the classical method of Nitsche [6], as well as the recent method of Oden, Babuska, and Baumann, see [7], and stabilized version thereof. The method of Nitsche has been analyzed by, among others, Wheeler [11], Arnold [1], and more recently in the context of domain decomposition by Becker, Hansbo, and Stenberg [3]. The method of Oden et al. is a nonsymmetric formulation for which the stability only has been analytically established for cubics and higher order polynomials in one spatial dimension, see [2]. However, extensive numerical results, presented in [7], indicate that the scheme is stable in one and two dimensions for quadratic and higher order approximations. The numerical experiments also indicated that the nonsymmetric scheme converged with order $p+1$ in L^2 for odd order of approximation p, but only p for even order. Optimal order energy norm error estimates for stabilized versions thereof, for which stability is trivial, have also been analyzed in [9] and [10].

Our analysis of these schemes build on a novel splitting of the discrete space into a direct sum of continuous piecewise polynomials and a space representing the discontinuous part of the functions also satisfying a special orthogonality relation. Based on this splitting we prove the following main results:

- A complete characterization of the parameters yielding an inf-sup stable scheme and, as a consequence, optimal order energy norm error estimates.
- An analysis of the error in L^2 explaining the different behaviour of the nonsymmetric scheme for odd and even order approximation.

Finally, we make some remarks on the elementwise conservative nature of the discontinuous Galerkin method, which is a common motivation for the use of these methods, see for instance [7] and [8]. We note that with the obvious definition of the numerical flux our family of methods is always elementwise conservative, but, the numerical flux is in fact nodally exact only for the symmetric scheme. For the nonsymmetric schemes there is a nonzero constant added to the flux. The nodal exactness of the flux corresponds to the well known nodal exactness of continuous Galerkin approximations of the Poisson equation in one dimension.

The paper is organized as follows: in Section 2 we present the model problem and method; in Section 3 we introduce the splitting of the discrete function space; in Section 4 we prove our main stability result and energy norm error estimates; in Section 5 we prove L^2 error estimates; and in Section 6 we study the convergence of the flux and the conservative nature of the method.

2 The model problem and method

2.1 The continuous equations

We consider the following one dimensional boundary value problem: find $u : \Omega \to \mathbf{R}$ such that

$$\begin{aligned} -u_{xx} &= f \quad \text{in } \Omega, \\ u &= g_D \quad \text{on } \Gamma_D, \\ u_n &= g_N \quad \text{on } \Gamma_N, \end{aligned} \tag{1}$$

where u_n denotes the normal derivative nu_x, with n the outward unit normal; the domain is $\Omega = [0, 1]$, with boundary $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$; and $f : \Omega \to \mathbf{R}$, $g_D : \Gamma_D \to \mathbf{R}$, and $g_N : \Gamma_N \to \mathbf{R}$ are given data. To insure uniqueness we shall assume that Γ_D is not empty.

2.2 The discontinuous Galerkin method

To formulate the discontinuous Galerkin method we let $\mathcal{K} = \{K_j\}$ be a partion of the interval Ω into N subintervals (elements) $K_j = [E_{j-1}, E_j]$ of length $h_{K_j} = E_j - E_{j-1}$, where $\mathcal{E} = \{E_j\}$, $0 = E_0 < E_1 < \cdots < E_N = 1$, is the set of nodes. At each node E we define a normal $n = n(E)$, such that $n(E_0) = -1$ and $n(E_j) = 1, j = 1, \ldots, N$. Furthermore, we let $\mathcal{V}^h$ be the space of discontinuous piecewise polynomials of degree p_K on each element K, i.e.,

$$\mathcal{V}^h = \{u : u|_K \in \mathcal{P}_{p_K}(K), K \in \mathcal{K}\}, \tag{2}$$

where $\mathcal{P}_{p_K}(K)$ is the space of polynomials of degree p_K on K.

Note that the length of the elements h_K as well as the order of the polynomials p_K may differ from element to element allowing h-p adaptivity.

The discontinuous Galerkin method reads: find $u^h \in \mathcal{V}^h$ such that

$$a(u^h, v) = l(v) \quad \text{for all } v \in \mathcal{V}^h, \tag{3}$$

where the bilinear form $a(\cdot, \cdot)$ is defined by

$$a(v, w) = a_{\mathcal{K}}(v, w) - a_{\mathcal{E}}(v, w) + \alpha a_{\mathcal{E}}(w, v) + \beta b_{\mathcal{E}}(v, w), \tag{4}$$

with

$$a_{\mathcal{K}}(v, w) = (v_x, w_x)_{\mathcal{K}}, \tag{5}$$

$$a_{\mathcal{E}}(v, w) = (\langle v_n \rangle, [w])_{\mathcal{E}_I} + (v_n, w)_{\mathcal{E}_D}, \tag{6}$$

$$b_{\mathcal{E}}(v, w) = (h^{-1}[v], [w])_{\mathcal{E}_I} + (h^{-1}v, w)_{\mathcal{E}_D}, \tag{7}$$

and the linear functional $l(\cdot)$ by

$$l(v) = (f, v) + (g_N, v)_{\mathcal{E}_N} + \alpha(v_n, g_D)_{\mathcal{E}_D} + \beta(h^{-1}g_D, v)_{\mathcal{E}_D}. \tag{8}$$

Here α is a a real parameter and β is a positive real parameter. Further we have used the following notations: the scalar products $(u,v)_{\mathcal{K}} = \sum_{K\in\mathcal{K}} \int_K uv dx$ and $(u,v)_A = \sum_{E\in A} u(E)v(E)$ with $A \subset \mathcal{E}$ a subset of nodes; the partion $\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N$, with $\mathcal{E}_I$ the set of interior nodes, $\mathcal{E}_D$ the nodes on the Dirichlet boundary Γ_D, and $\mathcal{E}_N$ the nodes on the Neumann boundary Γ_N; the nodal average $\langle u\rangle = (u^+ + u^-)/2$ and the nodal jump $[u] = u^- - u^+$, with $u^\pm = \lim_{s\to 0^+} u(E \mp sn(E))$; the normal derivative $u_n(E) = n(E)u_x(E)$, where $n(E)$ is the nodal normal defined above; and the meshsize h is the average mesh size $\langle h\rangle$ for interior nodes and $h/2$ on the boundary.

3 A two-scale formulation of the dG method

3.1 A splitting of $\mathcal{V}^h$

Key to our analysis is a splitting of $\mathcal{V}^h = \mathcal{V}_c^h \oplus \mathcal{V}_d^h$, into a direct sum of continuous functions $\mathcal{V}_c^h$ and discontinuous functions $\mathcal{V}_d^h$, satisfying a special orthogonality relation.

Theorem 3.1 *(a) There is a splitting of $\mathcal{V}^h$:*

$$\mathcal{V}^h = \mathcal{V}_c^h \oplus \mathcal{V}_d^h, \tag{9}$$

where $\mathcal{V}_c^h$ is the subspace of continuous functions

$$\mathcal{V}_c^h = \{v \in \mathcal{V}^h : v \in C(\Omega), v|_{\Gamma_D} = 0\} \tag{10}$$

and $\mathcal{V}_d^h \subset \mathcal{V}^h$ is defined by

$$\mathcal{V}_d^h = \{v \in \mathcal{V}^h : a_{\mathcal{K}}(w,v) + a_{\mathcal{E}}(w,v) = 0 \text{ for all } w \in \mathcal{V}^h\}. \tag{11}$$

(b) The following identities hold:

$$a(v_c, v_c) = a_{\mathcal{K}}(v_c, v_c), \tag{12}$$

$$a(v_c, v_d) = 0, \tag{13}$$

$$a(v_d, v_d) = \alpha a_{\mathcal{K}}(v_d, v_d) + \beta b_{\mathcal{E}}(v_d, v_d), \tag{14}$$

$$a(v_d, v_c) = (\alpha+1) a_{\mathcal{K}}(v_c, v_d) = (\alpha+1) a_{\mathcal{E}}(v_c, v_d), \tag{15}$$

for all $v_c \in \mathcal{V}_c^h$ and $v_d \in \mathcal{V}_d^h$.

Proof. To prove (a) we first show that $\mathcal{V}_c^h \cap \mathcal{V}_d^h = 0$. If $v \in \mathcal{V}_c^h \cap \mathcal{V}_d^h$ we have $a_{\mathcal{E}}(w,v) = 0$ and thus $a_{\mathcal{K}}(w,v) = 0$. Choosing $w = v$ gives $a_{\mathcal{K}}(v,v) = 0$ and thus v is a constant function, which together with the boundary condition $v = 0$ on Γ_D give $v = 0$.

Next we show that $\mathcal{V}^h = \mathcal{V}_c^h + \mathcal{V}_d^h$. For each $E \in \mathcal{E}_I \cup \mathcal{E}_D$ we solve the equation: find $\varphi_{d,E} \in \mathcal{V}^h$ such that

$$a_{\mathcal{K}}(w, \varphi_{d,E}) = \begin{cases} \langle w_n(E)\rangle & E \in \mathcal{E}_I, \\ w_n(E) & E \in \mathcal{E}_D, \end{cases} \quad \text{for all } w \in \mathcal{V}^h. \tag{16}$$

The solution $\varphi_{d,E}$ will have support in $\cup_{K\cap E\neq\emptyset}K$; be continuous in $\Omega\setminus E$; $[\varphi_{d,E}]=1$; and $\langle\varphi_{d,E}\rangle=0$. See the explicit formulas in Subsection 3.3 below. We now note that $v_c=v-v_d$, with

$$v_d=\sum_{E\in\mathcal{E}_I}\varphi_{d,E}[v(E)]+\sum_{E\in\mathcal{E}_D}\varphi_{d,E}v(E),$$

is indeed a continuous function which is zero on $\mathcal{E}_D$.

For (b) we have that $[v_c]=0$ and thus $a_{\mathcal{E}}(w,v_c)=b_{\mathcal{E}}(w,v_c)=0$ for all $w\in\mathcal{V}^h$ and $v_c\in\mathcal{V}_c^h$. Thus (12), $a(v_c,v_c)=a_{\mathcal{K}}(v_c,v_c)$, follows immediately. Next the definition of $\mathcal{V}_d^h$ gives

$$a(v_c,v_d)=a_{\mathcal{K}}(v_c,v_d)-a_{\mathcal{E}}(v_c,v_d)+\alpha a_{\mathcal{E}}(v_d,v_c)+\beta b_{\mathcal{E}}(v_d,v_c)=0,$$

which proves (13). Using that $a_{\mathcal{K}}(v_d,v_d)=a_{\mathcal{E}}(v_d,v_d)$ we get

$$\begin{aligned}a(v_d,v_d)&=a_{\mathcal{K}}(v_d,v_d)-a_{\mathcal{E}}(v_d,v_d)+\alpha a_{\mathcal{E}}(v_d,v_d)+\beta b_{\mathcal{E}}(v_d,v_c)\\&=\alpha a_{\mathcal{K}}(v_d,v_d)+\beta b_{\mathcal{E}}(v_d,v_d),\end{aligned}$$

which proves (14). Finally for (15) we have

$$\begin{aligned}a(u_d,v_c)&=a_{\mathcal{K}}(u_d,v_c)-a_{\mathcal{E}}(u_d,v_c)\\&\qquad+\alpha a_{\mathcal{E}}(v_c,u_d)+\beta b_{\mathcal{E}}(v_c,u_d)\\&=a_{\mathcal{K}}(v_c,u_d)+\alpha a_{\mathcal{E}}(v_c,u_d)\\&=(1+\alpha)a_{\mathcal{E}}(v_c,u_d)\\&=(1+\alpha)a_{\mathcal{K}}(u_d,v_c),\end{aligned}$$

where we used the symmetry of $a_{\mathcal{K}}(\cdot,\cdot)$. □

3.2 A two-scale formulation of the dG method

Writing $u^h=u_c^h\oplus u_d^h$ and using the identities (13) and (15) we write the variational equation (3) as a triangular system: find $u_c^h\oplus u_d^h\in\mathcal{V}_c^h\oplus\mathcal{V}_d^h$ such that

$$\begin{aligned}a(u_c,v_c)+(1+\alpha)a_{\mathcal{K}}(u_d,v_c)&=l(v_c),\\a(u_d,v_d)&=l(v_d).\end{aligned}\tag{17}$$

We note that with this particular splitting of $\mathcal{V}^h$ the discontinuous scales, $\mathcal{V}_d^h$, are in fact not coupled to the continuous scales, $\mathcal{V}_c^h$. Furthermore, in the symmetric case, when $\alpha=-1$, there is no coupling from the discontinuous scales to the continuous scales. The two scales are thus completely uncoupled, and the system (17) takes the diagonal form

$$\begin{aligned}a(u_c^h,v_c)&=l(v_c),\\a(u_d^h,v_d)&=l(v_d).\end{aligned}\tag{18}$$

3.3 Construction of a basis for $\mathcal{V}_d^h$

From the definition of $\mathcal{V}_d^h$ we derive the following equations for the basis functions

$$
\begin{gathered}
\varphi_{d,j}(E_{j-1}) = \varphi_{d,j}(E_{j+1}) = 0, \\
[\varphi_{d,j}(E_j)] = 1 \text{ and } \langle \varphi_{d,j}(E_j) \rangle = 0, \\
(\varphi_{d,j}, v)_K = 0 \text{ for all } v \in \mathcal{P}_{p_K-2}(K).
\end{gathered}
$$

Counting degrees of freedom we find that after the three first conditions are satisfied on each interval there remains $p_K - 1$ degrees of freedom and thus we can make sure that we have orthogonality to $p_K - 1$ functions corresponding to the dimension of the space of polynomials of order $p_K - 2$. Thus the above equations determine a basis for $\mathcal{V}_d^h$ uniquely.

Example 3.1 In Figure 1, we show the basis for $\mathcal{V}_d^h$ for uniform order of polynomials $p = 1, 2, 3, 4$. The functions are given by

$$
\begin{aligned}
\varphi_d(\xi) &= \frac{\xi \pm 1}{2}, && \xi \lessgtr 0, \quad p = 1, \\
\varphi_d(\xi) &= \frac{(\xi \pm 1)(1 \pm 3\xi)}{2}, && \xi \lessgtr 0, \quad p = 2, \\
\varphi_d(\xi) &= \frac{(\xi \pm 1)(1 \pm 8\xi + 10\xi^2)}{2}, && \xi \lessgtr 0, \quad p = 3, \\
\varphi_d(\xi) &= \frac{(\xi \pm 1)(1 \pm 15\xi + 45\xi^2 \pm 35\xi^3)}{2}, && \xi \lessgtr 0, \quad p = 4,
\end{aligned}
$$

where $\xi \in [-1, 1]$ and the mapping from $K_{j-1} \cup K_j = [E_{j-1}, E_{j+1}]$ onto $[-1, 1]$ is $\xi(x)|_{K_k} = (x - x_j)/h_k, k = j - 1, j$.

4 Error estimates in the energy norm

4.1 Norms and preliminaries

We let $\|v\|_{s,\omega}$ denote the standard Sobolev norms for $v \in H^s(\omega)$ on the set $\omega \subset \Omega$. For brevity we write $\|v\|_s = \|v\|_{s,\Omega}$, $\|v\|_\omega = \|v\|_{0,\omega}$, and $\|v\| = \|v\|_0$ for the L^2 norm. Next we define the energy norm for, $v \in \mathcal{V}^h$, by

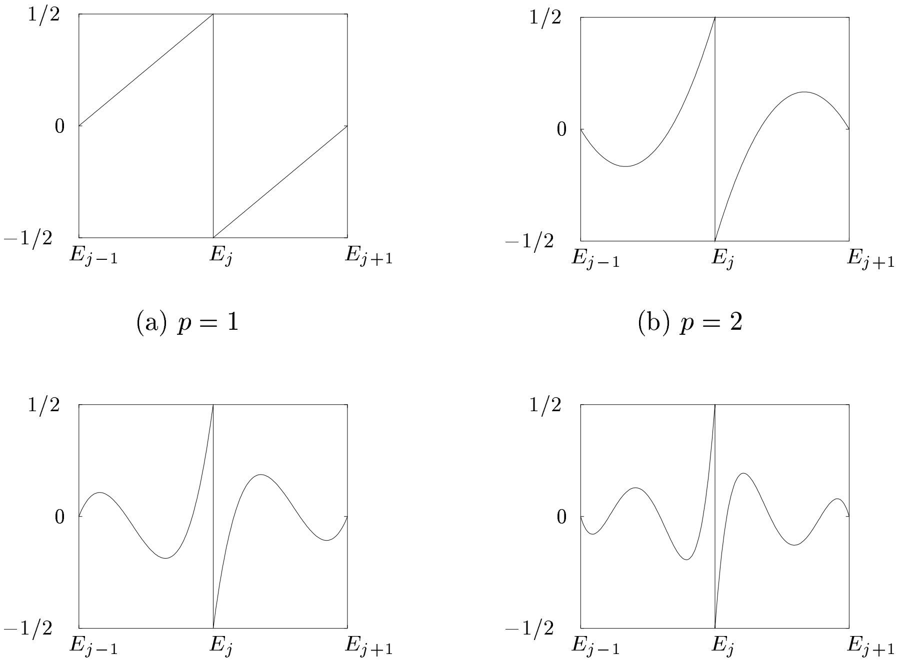
$$
|||v|||^2 = |||v|||_{\mathcal{K}}^2 + |||v|||_{\mathcal{E}}^2, \tag{19}
$$

where $|||v|||_{\mathcal{K}}$ and $|||v|||_{\mathcal{E}}$ are defined by

$$
|||v|||_{\mathcal{K}}^2 = a_{\mathcal{K}}(v, v) = (v_x, v_x)_{\mathcal{K}}, \tag{20}
$$

$$
|||v|||_{\mathcal{E}}^2 = b_{\mathcal{E}}(v, v) = (h^{-1}[v], [v])_{\mathcal{E}_I} + (h^{-1}v, v)_{\mathcal{E}_D}, \tag{21}
$$

where we used the notation introduced in Subsection 2.2.



(a) $p = 1$ (b) $p = 2$ (c) $p = 3$ (d) $p = 4$

Figure 1: Basis functions for $\mathcal{V}_d^h$ for $p = 1, \dots, 4$.

We now state some useful standard inequalities which we will need below. The following approximation property holds: for each $K \in \mathcal{K}$ there is a linear operator $\pi_K : H^1(K) \to \mathcal{P}_{p_K}(K)$ such that

$$\|u - \pi_K u\|_{l,K} \leq h_K^{k-l} \|u\|_{k,K} \quad \text{for all } v \in H^l(K),\ l = 0, 1. \tag{22}$$

where $\mathcal{V}^h(K) = \{v : v = w|_K, v \in \mathcal{V}^h\}$. A global interpolation operator $\pi : H^1 \to \mathcal{V}^h$ is defined by $(\pi v)|_K = \pi_K(v|_K)$. Further we have the trace inequality

$$\|v\|_{\partial K} \leq c \|v\|_K \|v\|_{1,K} \quad \text{for all } v \in H^1(K), \tag{23}$$

see [4], and the inverse estimate

$$\|v_n\|_{\partial K} \leq c h_K^{-1/2} \|v_x\|_K \quad \text{for all } v \in \mathcal{P}_{p_K}(K), \tag{24}$$

where h_K is the size of the element K. The inverse estimate is proved using scaling and the constant will depend on the degree of polynomials p. Finally, we mention the simple but useful inequality

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2, \tag{25}$$

for any a, b, and $\epsilon \in \mathbf{R}$ with $\epsilon > 0$.

4.2 Stability analysis

Our main result in this section is a complete characterization of the values of the parameters α and β for which the inf-sup constant, is positive and independent of the meshsize. This stability result is key for proving existence and energy norm error estimates.

We start with a result which gives a crucial equivalence between the edge and interior parts of the energy norm for functions in $\mathcal{V}_d^h$.

Lemma 4.1 *For $p \geq 1$ the following estimate holds*

$$|||v|||_{\mathcal{K}} \leq c_1 |||v|||_{\mathcal{E}} \quad v \in \mathcal{V}_d^h, \tag{26}$$

and, for $p \geq 2$ we also have the reverse estimate

$$|||v|||_{\mathcal{E}} \leq c_2 |||v|||_{\mathcal{K}} \quad v \in \mathcal{V}_d^h, \tag{27}$$

where c_1 and c_2 are constants independent of h but dependent on p.

Example 4.1 The constants c_1 and c_2 are independent of the meshsize h but depends on the degree of polynomials p used. Here we calculate approximate values of these constants in the case of $\Gamma_D = \Gamma$ for $p = 1, 2, 3, 4$. The values are summarized in the table below.

	p=1	p=2	p=3	p=4
c_1	1.00	3.00	6.00	10.00
c_2	∞	1.33	0.50	0.27

Note that $c_2 < \infty$ only for $p \geq 2$ since, depending on the boundary conditions, for $p = 1$ there may be piecewise constant functions in $\mathcal{V}_d^h$ forcing $c_2 = 0$ or there may exist sequences of functions such that $c_2 \to 0$ as $h \to 0$.

Proof. The first inequality follows by setting $w = v$ in (11) and the following estimates

$$\begin{aligned} |||v|||_{\mathcal{K}}^2 &= a_{\mathcal{K}}(v, v) \\ &= -a_{\mathcal{E}}(v, v) \\ &\leq c |||v|||_{\mathcal{K}} |||v|||_{\mathcal{E}}, \end{aligned}$$

where we used the inverse inequality (24). Finally dividing by $|||v|||$ yields the estimate. To prove the second estimate we consider an interior element K and assume that w is supported in K and use partial integration in (11) to obtain

$$a_{\mathcal{K}}(w, v) + a_{\mathcal{E}}(w, v) = (-w_{xx}, v)_K + ([w_x], \langle v \rangle)_{\partial K}.$$

Using the fact that $\langle v \rangle = 0$ we obtain $(-w_{xx}, v)_K = 0$. For $p \geq 2$ we may take w such that $-w_{xx} > 0$ and thus we conclude that v must be zero somewhere in K. The second inequality now follows immediately from scaling and finite dimensionality. $\square$

To prove the main result we first study the stability properties on $\mathcal{V}_c^h$ and $\mathcal{V}_d^h$. We sumarize our results in the following lemma.

Lemma 4.2 *(a) for all α, β there is a constant m_c, such that*

$$a(v_c, v_c) \geq m_c |||v_c|||^2 \quad \text{for all } v_c \in \mathcal{V}_c^h. \tag{28}$$

(b) If α, β satisfy one of the following inequalities:

$$\begin{gathered} \alpha, \beta \geq \delta \text{ for } p \geq 1, \\ \beta - c_1|\alpha| \geq \delta \text{ for } p \geq 1, \\ \alpha - c_2|\beta| \geq \delta \text{ for } p \geq 2, \end{gathered}$$

for some positive constant $\delta > 0$. Then there is a constant $m_d > 0$, such that

$$a(v_d, v_d) \geq m_d |||v_d|||^2 \quad \text{for all } v_d \in \mathcal{V}_d^h. \tag{29}$$

The constants c_1, and c_2 are defined in Lemma 4.1.

Proof. To prove *(a)*, we use (12) and the fact that $|||v_c|||_{\mathcal{E}} = 0$ to get

$$a(v_c, v_c) = a_{\mathcal{K}}(v_c, v_c) = |||v_c|||_{\mathcal{K}}^2 = |||v_c|||^2.$$

Hence *(a)* follows with $m_c = 1$.

Next we turn to *(b)*. Using (14) we obtain

$$\begin{aligned} a(v_d, v_d) &= \alpha a_{\mathcal{K}}(v_d, v_d) + \beta b_{\mathcal{E}}(v_d, v_d) \\ &= \alpha |||v_d|||_{\mathcal{K}}^2 + \beta |||v_d|||_{\mathcal{E}}^2. \end{aligned}$$

For $\alpha, \beta \geq \delta$ we get (29) with $m_d = \delta$. Next for $\alpha \leq 0$, we obtain

$$\begin{aligned} a(v_d, v_d) &= \alpha |||v_d|||_{\mathcal{K}}^2 + \beta |||v_d|||_{\mathcal{E}}^2 \\ &= m_d |||v_d|||^2 + (\alpha - m_d)|||v_d|||_{\mathcal{K}}^2 + (\beta - m_d)|||v_d|||_{\mathcal{E}}^2 \\ &\geq m_d |||v_d|||^2 + ((\beta - m_d) - c_1(|\alpha| + m_d))|||v_d|||_{\mathcal{E}}^2, \end{aligned}$$

where we used (26). Thus for $p \geq 1$ and α, β such that $\beta - c_1|\alpha| \geq \delta > 0$, we get (29) with $m_d = \delta/(1 + c_1)$.

Finally, for $p \geq 2$, we instead use (27) to get

$$\begin{aligned} a(v_d, v_d) &= \alpha |||v_d|||_{\mathcal{K}}^2 + \beta |||v_d|||_{\mathcal{E}}^2 \\ &= m_d |||v_d|||^2 + (\alpha - m_d)|||v_d|||_{\mathcal{K}}^2 + (\beta - m_d)|||v_d|||_{\mathcal{E}}^2 \\ &\geq m_d |||v_d|||^2 + ((\alpha - m_d) - c_2(|\beta| + m_d))|||v_d|||_{\mathcal{K}}^2 \end{aligned}$$

Thus for $p \geq 2$, and α, β such that $\alpha - c_2|\beta| \geq \delta > 0$, we get (29) with $m_d = \delta/(1 + c_2)$. $\square$

Theorem 4.1 *If the assumptions in Lemma 4.2 hold. Then there is a constant $m > 0$, independent of h, such that*

$$\inf_{u \in V} \sup_{v \in V} \frac{a(u,v)}{|||u|||\,|||v|||} \geq m.$$

Proof. Writing $u = u_c + u_d$ and $v = v_c + v_d$, with $u_c, v_c \in \mathcal{V}_c^h$, and $u_d, v_d \in \mathcal{V}_d^h$, we have

$$a(u_c + u_d, v_c + v_d) = a(u_c, v_c) + (\alpha + 1)a_{\mathcal{K}}(u_c, v_d) + a(u_d, v_d),$$

where we used (13) and (15). Next setting

$$v_c + v_d = u_c + \gamma u_d, \tag{30}$$

where $\gamma \in \mathbf{R}$ is a parameter, we get

$$\begin{aligned} a(u_c + u_d, v_c + v_d) &= a(u_c, u_c) + (\alpha + 1)a_{\mathcal{K}}(u_d, u_c) + \gamma a(u_d, u_d). \\ &\geq m_c|||u_c|||^2 - |1 + \alpha|\,|||u_d|||\,|||u_c||| + m_d\gamma|||u_d|||^2 \\ &\geq (m_c - |1 + \alpha|\epsilon)|||u_c|||^2 + (m_c\gamma - |1 + \alpha|\epsilon^{-1})|||u_d|||^2. \end{aligned}$$

Here we used Lemma 4.2, the Cauchy-Schwarz inequality, and finally (25). Choosing ϵ such that $m_c - (1 + \alpha)\epsilon \geq m'/2$ and $\gamma \geq 1$ such that $m_c\gamma - (1 + \alpha)\epsilon^{-1} \geq m'/2$, we get

$$a(u_c + u_d, v_c + v_d) \geq \frac{m'}{2}\Big(|||u_c|||^2 + |||u_d|||^2\Big). \tag{31}$$

for $v_c + v_d$ defined in (30). Next we note that, for $\gamma \geq 1$, we have

$$\frac{|||u_c|||^2 + |||u_d|||^2}{|||u_c||| + \gamma|||u_d|||} \geq \frac{1}{2\gamma}\Big(|||u_c||| + |||u_d|||\Big) \geq \frac{1}{2\gamma}|||u_c + u_d|||, \tag{32}$$

where the first inequality is proved using (25) and the second is just the triangle inequality. Combining (31) and (32) we immediately get the desired inf-sup bound

$$\inf_{u \in \mathcal{V}^h} \sup_{v \in \mathcal{V}^h} \frac{a(u,v)}{|||u|||\,|||v|||} \geq \frac{m'}{4\gamma} = m.$$

$\square$

4.3 Computations of the inf-sup constant

In this section we verify our analytical results by numerical computations of the inf-sup constant for various values of the parameters α and β, the meshsize h, and the order of polynomials p.

The inf-sup constant can be calculated as follows

$$m = \sqrt{\lambda_{\min}}, \tag{33}$$

where $\lambda_{\min}$ is the smallest eigenvalue of the eigenvalue problem

$$Ax = \lambda Bx. \tag{34}$$

Here B is the matrix associated with the energy norm, i.e., $|||v|||^2 = \hat{v}^T B\hat{v}$, $A = K^T B^{-1} K$ with K the stiffness matrix, i.e., $a(v, w) = \hat{v}^T K\hat{w}$, and $\hat{v}$ denotes a coordinate representation of $v \in \mathcal{V}^h$. See [7] for details on the derivation of the eigenvalue problem (34).

Example 4.2 We calculate the inf-sup constant $m(\alpha, \beta)$ as a function of $\alpha \in [-2, 2]$ and $\beta \in [-2, 10]$. We also demand that the discrete operator has no eigenvalues in the left half-plane. Whenever there is an eigenvalue in the left hand half plane we define $m = 0$. Further we consider the case of homogeneous Dirichlet conditions on the boundary and we discretize with uniform partion of Ω into 10 elements and polynomials of degree $p = 1, \dots, 3$. Results are presented in Figure 2.

4.4 Energy norm error estimates

Based on the inf-sup stability analysis presented above we can now derive optimal order energy norm error estimates.

Theorem 4.2 *Under the assumptions on α and β in 4.2, the following energy norm error estimate holds*

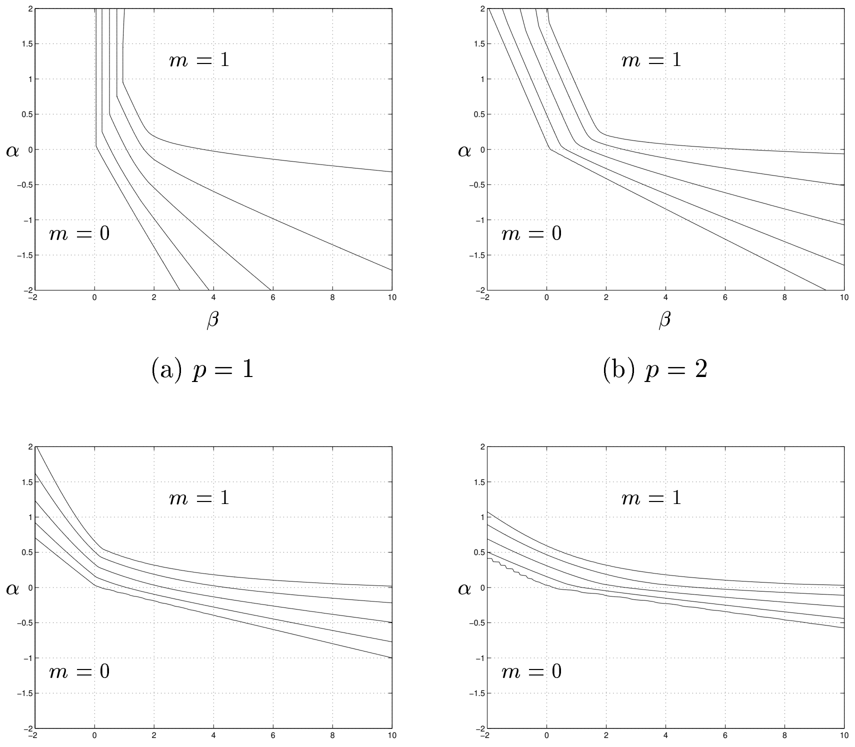
$$|||u - u^h||| \le c\Big(1 + \frac{c}{m}\Big)\left(\sum_{K\in\mathcal{K}} {h_K}^{2p_K} \|u\|^2_{p_K+1,K}\right)^{1/2}.$$

Proof. First we add and subtract the interpolant πu of u to get $u - u^h = u - \pi u + \pi u - u^h$. Here the first term is easily estimated:

$$|||u - \pi u||| \le c\left(\sum_{K\in\mathcal{K}} {h_K}^{2p_K} \|u\|^2_{p_K+1,K}\right)^{1/2}. \tag{35}$$

using the trace inequality (23) followed by the approximation property (22). Next for the second term it follows from Theorem 4.1 that

$$|||\pi u - u^h||| \le \frac{1}{m} \sup_{v\in\mathcal{V}^h} \frac{a(\pi u - u^h, v)}{|||v|||}. \tag{36}$$



(a) $p = 1$ (b) $p = 2$

(c) $p = 3$ (d) $p = 4$

Figure 2: Contours corresponding to the following values of the inf-sup constant $m = 0.05, 0.25, 0.5, 0.75, 0.95$, from left to right, for $p = 1, 2, 3, 4$.

Using Galerkin orthogonality (3) we obtain $a(\pi u - u^h, v) = a(\pi u - u, v)$. Since $v \in \mathcal{V}^h$, we may apply the inverse trace inequality (24) to get the following estimate

$$|a_{\mathcal{K}}(\pi u - u, v)| + \alpha |a_{\mathcal{E}}(v, \pi u - u)| + \beta |b_{\mathcal{E}}(\pi u - u, v)| \leq c|||u - \pi u|||\,|||v|||. \tag{37}$$

Furthermore, using the trace inequality (23) we obtain

$$|a_{\mathcal{E}}(\pi u - u, v)| \leq c \left(\sum_{K \in \mathcal{K}} h_K \|u - \pi u\|_K \|u - \pi u\|_{1,K} \right)^{1/2} |||v|||. \tag{38}$$

Combining (36), (37), and (38), and finally using the approximation property (22) we obtain

$$|||\pi u - u^h||| \leq \frac{c}{m} \left(\sum_{K \in \mathcal{K}} {h_K}^{2p_K} \|u\|^2_{p_K+1,K} \right)^{1/2}, \tag{39}$$

which together with (35) proves the theorem. $\square$

5 Error estimates in the L^2 norm

In the remainder of the paper we assume that the a uniform mesh with parameter h and order of polynomials p.

5.1 Preliminary estimates

Introducing the continuous dual problem: find $\phi \in \mathcal{V}$ such that

$$(v, e) = a_{\mathcal{K}}(v, \phi) - a_{\mathcal{E}}(\phi, v) \quad \text{for all } v \in \mathcal{V}, \tag{40}$$

where $e = u - u^h \in L^2(\Omega)$ is the error. We note that ϕ will be the solution of the Poisson equation. Setting $v = e$ and using that $[\phi] = 0$ we get the error representation formula

$$\begin{aligned} \|e\|^2 &= a(e, \phi) - (\alpha + 1) a_{\mathcal{E}}(\phi, e) \\ &= a(e, \phi - \pi\phi) - (\alpha + 1) a_{\mathcal{E}}(\phi, e), \end{aligned} \tag{41}$$

where we used the Galerkin orthogonality (3) to subtract an interpolant $\pi\phi \in \mathcal{V}^h$ of ϕ. Note that we cannot subtract an interpolant from the last term.

Using the Cauchy-Schwarz inequality and the approximation property (22) for the first term and observing that $a_{\mathcal{E}}(\phi, e) = -a_{\mathcal{E}}(\phi, u_d)$ and using the Cauchy-Schwarz followed by the trace inequality (23) for the second we obtain

$$\|e\|^2 \leq Ch|||e|||\, \|\phi\|_2 + |(\alpha + 1)|\, |||\phi|||_1^{1/2} \|\phi\|_2^{1/2} |||u_d|||.$$

Finally, invoking the standard regularity estimates

$$\|\phi\|_1 \leq C\|e\|, \quad \|\phi\|_2 \leq C\|e\|, \tag{42}$$

we get

$$\|e\| \leq C\Big(h|||e||| + |(\alpha + 1)|\, |||u_d|||\Big). \tag{43}$$

Note that the important part is the second term which vanish for the symmetric formulation but gives a nonzero contribution for $\alpha \neq 1$. In the symmetric case, $\alpha = -1$, we immediately get the standard L^2 error estimate

$$\|e\| \leq Ch^{p+1} \|u\|_{p+1}, \tag{44}$$

by using (4.2). We next turn to an estimate of the remaining term occurring for $\alpha \neq -1$.

5.2 The nonsymmetric case: uniform mesh

We shall now prove an L^2 error estimate for the nonsymmetric case, which gives an improved rate of convergence for odd order approximation provided the solution enjoys some additional regularity assumptions. We start with an improved estimate of the energy norm of the discontinuous part of the solution.

Lemma 5.1 *If the assumptions in Lemma 4.2 part (b) holds and the mesh is uniform. Then we have the estimate*

$$|||u_d^h||| \leq ch^{p+\beta(s)}\|u\|_{p+1+s}, \tag{45}$$

where $0 \leq s \leq 1$*, and the function* $\beta(s)$ *is defined by*

$$\beta(s) = \begin{cases} s & p \text{ odd,} \\ 0 & p \text{ even.} \end{cases} \tag{46}$$

Proof. By Lemma 4.2 and the equation for u_d^h we get

$$\begin{aligned} m_d|||u_d^h|||^2 &\leq a(u_d^h, u_d^h) \\ &= l(u_d^h) \\ &\leq \sum_{E\in\mathcal{E}} \hat{u}_{d,E}^h |(f, \varphi_{d,E})|, \end{aligned}$$

where we used the expansion $u_d^h = \sum_{E\in\mathcal{E}} \hat{u}_{d,E}^h \varphi_{d,E}$.

We shall now study the orthogonality properties of $\varphi_{d,E}$. Let $L_E = \text{supp}(\varphi_{d,E})$. First we observe that $\varphi_{d,E}$ is orthogonal to discontinuous piecewise polynomials of order $p-2$, thus in particular to polynomials of order $p-2$ on L_E. Furthermore, for odd p we note that $\varphi_{d,E}$ is also orthogonal to $(x-x_E)^{p-1}$. This extra orthogonality holds since when the mesh is uniform $\varphi_{d,E}$ is an odd function and for odd p, $(x-x_E)^{p-1}$ is even.

Thus for odd p we have

$$(f, \varphi_{d,E}) = (f - \Pi_L f, \varphi_{d,E}),$$

where Π_L denotes an interpolation operator on L onto polynomials of order $p-1$ for odd p and $p-2$ for even p.

Using the Cauchy-Schwarz inequality followed by a variant of the approximation property (22) we get

$$\begin{aligned} |(f, \varphi_{d,E})| &= |(f - \Pi_L f, \varphi_{d,E})| \\ &\leq \|f - \Pi_L f\|_L \, \|\varphi_{d,E}\|_L \\ &\leq ch^{p-1+\beta(s)}\|f\|_{s,L}\|\varphi_{d,E}\|_L. \end{aligned}$$

Collecting the estimates we get

$$\begin{aligned} |(f, u_d^h)| &= \sum_{E\in\mathcal{E}} ch^{p-1+\beta(s)}\|f\|_{s,L}\hat{u}_{d,E}^h\|\varphi_{d,E}\|_L \\ &\leq c\Big(\sum_{E\in\mathcal{E}} h^{2(p-1+\beta(s))}\|f\|_{s,L}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}}(\hat{u}_{d,E}^h)^2\|\varphi_{d,E}\|_L^2\Big)^{1/2} \\ &\leq ch^{p+\beta(s)}\|f\|_s|||u_d^h|||, \end{aligned}$$

where at last we used the estimate

$$\sum_{E\in\mathcal{E}}(\hat{u}^h_{d,E})^2\|\varphi_{d,E}\|^2_L \leq c\bigg(\sum_{E\in\mathcal{E}} h(\hat{u}^h_{d,E})^2[\varphi_{d,E}(E)]^2\bigg)$$
$$\leq ch^2\bigg(\sum_{E\in\mathcal{E}} h^{-1}[u^h_d(E)]^2\bigg)$$
$$\leq ch^2|||u^h_d|||^2_{\mathcal{E}}.$$

Finally dividing by $|||u^h_d|||$ yields the desired result. □

Using this lemma we immediately obtain the following L^2 norm error estimate.

Theorem 5.1 *If the assumptions in Lemma 4.2 (b) holds and the mesh is uniform. Then we have the following error estimate*

$$\|e\| \leq C\Big(h^{p+1}\|u\|_{p+1} + |\alpha+1|h^{p+\beta(s)}\|u\|_{p+1+s}\Big), \tag{47}$$

where $\beta(s)$ is defined in Lemma 5.1.

Proof. Combining the L^2 error estimate (43) with the energy norm error estimate in Theorem (4.2) and Lemma 5.1 gives the estimate. □

Example 5.1 We calculate the rate of convergence as a function of $\alpha \in [-2, 2]$, by using a least squares fit on numerically computed L^2 errors obtained from uniform meshes with $25, 50$, and 100, elements for $p = 1, \ldots, 4$. To insure a stable scheme for all $\alpha \in [-2, 2]$ we have chosen $\beta = 8, 15, 40, 80$, for $p = 1, 2, 3, 4$, respectively. The underlying equation is (1) with $f = sin(\pi x)/\pi^2$ and homogeneous Dirichlet conditions. We thus have $u \in H^s$ for any s and Theorem 5.1 predicts that we will obtain convergence of order $p + 1$ for odd p and all α, and order p for even p and $\alpha \neq -1$, for which we expect order $p + 1$. The numerical results presented in Figure 3 confirmes the prediction.

6 Estimates of the error in the flux

Introducing the numerical flux

$$\sigma^h_n(E) = \langle u_n(E)\rangle - \beta h^{-1}[u(E)], \quad E \in \mathcal{E}_I \cup \mathcal{E}_D. \tag{48}$$

We observe that for each element $K \in \mathcal{K}$ the conservation law

$$\int_{\partial K\setminus\Gamma_N} \sigma^h_n + \int_{\partial K\cap\Gamma_N} g_N + \int_K f = 0, \tag{49}$$

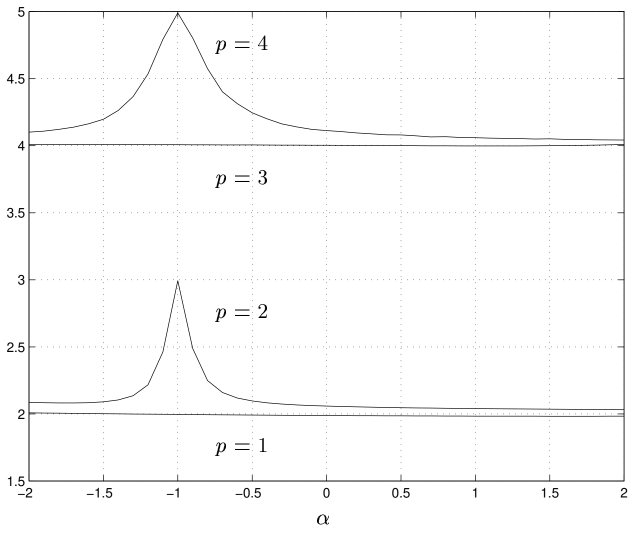


Figure 3: The convergence in the L^2 norm for $p = 1, \ldots, 4$ for $-2 \leq \alpha \leq 2$.

holds, by choosing the test function in (3) to be one on K and zero elsewhere.

We shall now derive an estimate for the error in the numerical flux $\sigma_n^h(u^h) - \sigma_n(u)$, where $\sigma_n(u) = u_n$. Note first that if Γ_N is non empty then it follows from (48) that $\sigma_n^h = \sigma_n$, i.e., the flux is exact. In the case of $\Gamma_D = \Gamma$ the same reasoning shows that

$$\sigma_n(u) - \sigma_n^h(u^h) = c, \tag{50}$$

for some constant c, i.e., the error must be constant.

We first observe that given ψ there is a unique $v \in \mathcal{V}^h$ such that $[v] = \psi, [v_x] = 0, v_{xx}|_K = 0$. We then have

$$a_{\mathcal{K}}(w, v) - a_{\mathcal{E}}(v, w) = 0, \tag{51}$$

for $w = u$ and any $w \in \mathcal{V}^h$. Using this identity we conclude that

$$\begin{aligned}(\sigma_n^h(w), \psi) &= a_{\mathcal{E}}(w, v) - \beta b(w, v) \\ &= a(w, v) + (1 + \alpha) a_{\mathcal{E}}(v, w).\end{aligned} \tag{52}$$

Furthermore, setting $w = u - u^h$ we get

$$\begin{aligned}(\sigma_n(u) - \sigma_n^h(u^h), \psi) &= a(u - u^h, v) + (1 + \alpha) a_{\mathcal{E}}(v, u - u^h), \\ &= (1 + \alpha) a_{\mathcal{E}}(v, u - u^h),\end{aligned} \tag{53}$$

since $a(u - u^h, v) = 0$ for $v \in \mathcal{V}^h$ by Galerkin orthogonality (3). Finally, using that $a_{\mathcal{E}}(v, u - u^h) = -a_{\mathcal{E}}(v, u_d^h)$ we obtain the error representation formula

$$(\sigma_n(u) - \sigma_n^h(u^h), \psi) = -(1 + \alpha) a_{\mathcal{E}}(v, u_d^h). \tag{54}$$

Using our earlier estimates we obtain the following error estimate.

Theorem 6.1 *If the assumptions in Lemma 4.2 (b) holds and the mesh is uniform. Then we have the following error estimate*

$$\left(\sum_{E\in\mathcal{E}} h\Big(\sigma_n(u) - \sigma_n^h(u^h)\Big)^2\right)^{1/2} \leq c|\alpha+1|h^{p+1+\beta(s)}\|u\|_{p+1+s}, \tag{55}$$

where $\beta(s)$ is defined in Lemma 5.1.

Proof. Starting from the error representation formula we have

$$\begin{aligned} |(\sigma_n(u) - \sigma_n^h(u^h), \psi)| &\leq |1+\alpha|\,|a_{\mathcal{E}}(u_d^h)| \\ &\leq |1+\alpha|\,|\langle v_x(E)\rangle|\,|[u_d^h(E)]| \\ &\leq C|1+\alpha|\,|[u_d^h(E)]|, \end{aligned}$$

since $|\langle v_x(E)\rangle| \leq C$. Combining this estimate with Lemma 5.1 we obtain the desired result. $\square$

Thus we note that for the symmetric case we have $\alpha = -1$ and the error is identically zero, while in the nonsymmetric case the error will not be zero but it will have high order of convergence.

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