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Analysis of a Nonsymmetric Discontinuous Galerkin Method for Elliptic Problems: Stability and Energy Error Estimates*

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Abstract

In this paper we analyze a nonsymmetric discontinuous Galerkin method for elliptic problems proposed by Oden, Babuška, and Baumann. Our main results are a complete inf-sup stability analysis and, as a consequence, error estimates in a mesh dependent energy norm allowing variable meshsize and order of polynomials. The analysis is carried out in two spatial dimensions on an unstructured triangulation.

1 Introduction

Discontinuous Galerkin (dG) methods for numerical approximation of partial differential equations is a classical technique which have recently received new interest, motivated by some attractive features including a flexible discretization allowing easy implementation of h-p adativity , nonmatching grids, and a local conservation property. Of course there are disadvantages too, the number of degrees of freedom is larger, see [12], and efficient iterative solvers are not yet developed.

In this paper we are concerned with the analytical and numerical study of the recent nonsymmetric dG method for elliptic problems proposed by Oden, Babuška, and Baumann in [16]. This method does not contain the stabilizing (penalty) term as the classical symmetric Nitsche method [15]. Plenty of numerical results were presented in [16], showing that a remarkable stability is hidden in the nonsymmetric form for polynomials of order higher or equal to two in one and two spatial dimensions. The desire to analytically understand the stability properties of the nonsymmetric dG method is the motivation for the

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present paper. In an earlier paper [13] Larson and Niklasson showed complete stability estimates for a family of dG methods, including both the nonsymmetric method and the symmetric Nitsche method, in one spatial dimension. These results extended the analytical stability estimates presented by Babuška, Baumann, and Oden in [4] for polynomials of order three or higher in one spatial dimensions. The analysis presented in this paper builds on the ideas in [13].

Our main result in this work is a complete discrete stability analysis, where we prove that the method is inf-sup stable with respect to a mesh dependent energy norm for quadratic and higher order polynomials on a general unstructured triangulation in two spatial dimensions. We present numerical calculations of the inf-sup constant confirming our analytical estimates. Our analytical and numerical results confirms the numerical observations reported in [16]. The case of linear ploynomials is also investigated and we show that the inf-sup constant is either zero or depends on the meshsize (depending on boundary conditions) if the mesh is of checkerbord type.

From the study of the discrete stability properties we immediately obtain optimal order a priori error estimates in the energy norm, allowing, local meshsize as well as local degree of polynomials. We present numerical results illustrating our error estimates. In two recent papers, Riviére, Wheeler, and Girault [17] and [18], prove an a priori error estimate of the L^2 norm of the gradient of the error for the nonsymmetric dG method by relating it to a method where the discontinuities on each edge have average zero. However, no stability estimate for the nonsymmetric dG method is presented. We also mention the comprehensive overview and analysis of a large class of dG methods by Arnold, Brezzi, Cockburn, and Marini [3].

Key to our analysis is a splitting of the space of all discontinuous piecewise polynomials into a sum of a space of functions with constrained discontinuities, representing continuous scales, and a space of discontinuous functions with small spatial mean value. This splitting, properly constructed, leads to a triangular system which can be analyzed.

The dG method for elliptic problems appears to originate from the work of Nitsche[15], where a consistent weak treatment of Dirichlet boundary conditions was introduced. Later methods based on discontinuous approximation and weak enforcement of continuity om interelement boundaries by means of terms similar to Nitsches method were introduced, see Douglas and Dupont [10], Baker [6], Wheeler [20], and Arnold [2]. Recently the interest for dG methods have increased, see the proceedings [9], in part, motivated by the success of dG methods for hyperbolic problems. Furthermore, extensions of dG methods to second order problems have been suggested by Bassi and Rebay [7] and Oden et al. [16]. See also Arnold et al. [3] for an overview.

The remainder of this paper is organized as follows: in Section 1 we introduce the nonsymmetric dG method and the necessary notation; in Section 2 we introduce the nonsymmetric dG method; in Section 3 we present the splitting of the discontinuous piecewise polynomial space and the two scale formulation of the dG method; and finally in Section 4 we show the stability estimate and the error estimate in the energy norm.

2 The model problem and dG method

2.1 A model problem

Let Ω be a polygonal domain in $\mathbf{R}^2$ with boundary Γ divided into two disjoint parts $\Gamma = \Gamma_N \cup \Gamma_D$. We consider the following linear elliptic model problem: find $u : \Omega \to \mathbf{R}$ such that

$$\begin{aligned} -\nabla \cdot \sigma(u) &= f \quad && \text{in } \Omega, \\ u &= g_D \quad && \text{on } \Gamma_D, \\ \sigma_n(u) &= g_N \quad && \text{on } \Gamma_N. \end{aligned} \tag{2.1}$$

Here the flux $\sigma(u)$ is defined by

$$\sigma(u) = A\nabla u, \tag{2.2}$$

with A a constant (or piecewise constant) symmetric positive definite matrix and $\sigma_n(u)$ denotes the normal flux

$$\sigma_n(u) = n \cdot A\nabla u, \tag{2.3}$$

where n is the exterior unit normal of Γ. It is well known that there is a unique solution in $H^1(\Omega)$ for $f \in H^{-1}(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in H^{-1/2}(\Gamma_N)$ to (2.1), see [11], where $H^s(\omega)$ denote the standard Sobolev spaces on the set ω.

2.2 Discrete spaces

We let $\mathcal{K}$ be a triangulation of Ω into shape regular triangles K and we denote the set of all edges E by $\mathcal{E}$. Further the set of edges is divided into three disjoint sets

$$\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N, \tag{2.4}$$

where $\mathcal{E}_I$ is the set of all edges in the interior of Ω, $\mathcal{E}_D$ the edges on the Dirichlet part of the boundary Γ_D, and $\mathcal{E}_N$ the edges on the Neumann part Γ_N. We let $h : \Omega \to \mathbf{R}$ denote the mesh function such that $h|_K = h_K = \text{diam}(K)$ and $h|_E = h_E = \text{diam}(E)$, i.e., the length of the edge E. We let

$$\mathcal{V} = \bigoplus_{K \in \mathcal{K}} \mathcal{P}_p(K), \tag{2.5}$$

where $\mathcal{P}_p(K)$ is the space of all polynomials of degree less or equal to p defined on K. The degree of polynomials, as well as the meshsize, may vary from element to element so that $p|_K = p_K$, and thus we allow h-p adaptivity.

2.3 The nonsymmetric dG method

In [16] Oden, Babuška, and Baumann proposed the following nonsymmetric dG method: find $u_h \in \mathcal{V}$ such that

$$a(u_h, v) = l(v) \quad \text{for all } v \in \mathcal{V}. \tag{2.6}$$

Here $a(\cdot, \cdot)$ is a bilinear form defined by

$$a(v, w) = a_{\mathcal{K}}(v, w) - a_{\mathcal{E}}(v, w) + a_{\mathcal{E}}(w, v), \tag{2.7}$$

where

$$a_{\mathcal{K}}(v, w) = \sum_{K \in \mathcal{K}} (\sigma(v), \nabla w)_K, \tag{2.8}$$

$$a_{\mathcal{E}}(v, w) = \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(v) \rangle, [w])_E, \tag{2.9}$$

and $l(\cdot)$ is a linear functional defined by

$$l(v) = (f, v) + \sum_{E \in \mathcal{E}_N} (g_N, [v])_E + \sum_{E \in \mathcal{E}_D} (g_D, \langle \sigma_n(v) \rangle)_E. \tag{2.10}$$

We employed the notation

$$\langle v \rangle = \begin{cases} (v^+ + v^-)/2 & E \in \mathcal{E}_I, \\ v^+ & E \in \mathcal{E}_D, \end{cases} \tag{2.11}$$

for the average and

$$[v] = \begin{cases} v^+ - v^- & E \in \mathcal{E}_I, \\ v^+ & E \in \mathcal{E}_D, \end{cases} \tag{2.12}$$

for the jump at an edge E, where $u^\pm(x) = \lim_{t \to 0, t > 0} u(x \mp tn)$, $x \in E$, and n is the exterior unit normal to E for $E \in \mathcal{E}_D \cup \mathcal{E}_N$ and a fixed, but arbitrary, unit normal to E for $E \in \mathcal{E}_I$, see Figure 1.

Lemma 2.1 *If $f \in L^2$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in L^2(\Gamma_N)$ then the linear functional $l(\cdot)$ is bounded on $\mathcal{V}$ and the exact solution u of (2.1) satisfies*

$$a(u, v) = l(v), \tag{2.13}$$

for all $v \in \mathcal{V}$.

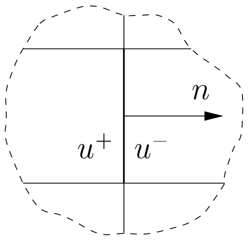


Figure 1: The plus and minus sides of an edge.

Proof. The first statement is obvious. For the second we note that the normal trace $\sigma_n(u)$ of $\sigma(u)$ is well defined in $L^2(E)$ on all edges $E \in \mathcal{E}$ since the stability estimate $\|\sigma(u)\| + \|\nabla \cdot \sigma(u)\| \leq c(\|f\| + \|g_D\|_{1/2,\Gamma_D} + \|g_N\|_{\Gamma_N})$ holds. $\square$

Here and below we let $\|v\|_{s,\omega}$ and $|v|_{s,\omega}$, denote the standard Sobolev norms and semi-norms, respectively, for $v \in H^s(\omega)$ on the set $\omega \subset \Omega$. For brevity we write $\|v\|_s = \|v\|_{s,\Omega}$, $\|v\|_\omega = \|v\|_{0,\omega}$, and $\|v\| = \|v\|_{0,\Omega}$ for the L^2 norm.

Remark 2.1 The dG method enjoys a local elementwise conservation property. Restricting, for simplicity, our attention to an element K such that $\partial K \cap \Gamma = \phi$ we obtain the discrete conservation law

$$\int_K f + \int_{\partial K} \langle \sigma_n(u) \rangle = 0, \tag{2.14}$$

by choosing $v = 1$ on K and $v = 0$ on $\Omega \setminus K$ in (2.6). See, the discussion in [16].

2.4 The energy norm and some useful inequalities

We equip $\mathcal{V}$ with the following mesh dependent energy norm

$$|||v|||^2 = |||v|||_{\mathcal{K}}^2 + \|\langle \sigma_n(v) \rangle\|_{\mathcal{E}}^2 + \|h^{-1}[v]\|_{\mathcal{E}}^2, \tag{2.15}$$

where

$$|||v|||_{\mathcal{K}}^2 = \sum_{K \in \mathcal{K}} (A\nabla v, \nabla v)_K, \tag{2.16}$$

$$\|w\|_{\mathcal{E}}^2 = \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} \|h^{1/2} w\|_E^2. \tag{2.17}$$

Next we recall some useful standard inequalities which we will need in our developments. First we have the trace inequality

$$\|v\|_{\partial K}^2 = c\|v\|_K \Big(h_K^{-1} \|v\|_K + \|v\|_{1,K} \Big) \quad \text{for } v \in H^1(K), \tag{2.18}$$

where c is a constant independent of h. This inequality follows by mapping to the unit size reference element $\tilde{K}$, employing the trace inequality

$$\|v\|_{\partial\tilde{K}}^2 \leq c\|v\|_{\tilde{K}}\|v\|_{1,\tilde{K}} \quad \text{for } v \in H^1(\tilde{K}), \tag{2.19}$$

see Brenner and Scott [8], and finally transforming back to K. Furthermore, the following inverse estimate will be useful

$$\|\langle\sigma_n(v)\rangle\|_{\mathcal{E}} \leq C|||v|||_{\mathcal{K}} \quad \text{for } v \in \mathcal{V}, \tag{2.20}$$

with constant C dependent on the degree of polynomials p but not on the meshsize h. This estimate can be shown by scaling, see Thomée [19] for details. Finally we mention the useful inequality

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2, \tag{2.21}$$

for any a, b, and $\epsilon \in \mathbf{R}$ with $\epsilon > 0$.

3 A two-scale formulation of the dG method

3.1 A splitting of $\mathcal{V}$

Theorem 3.1 *For $p \geq 2$ there is a decomposition of $\mathcal{V}$ into a direct sum*

$$\mathcal{V} = \mathcal{V}_c + \mathcal{V}_d, \tag{3.1}$$

where

$$\mathcal{V}_d = \{v \in \mathcal{V} : a_{\mathcal{K}}(w,v) - a_{\mathcal{E}}(w,v) = 0 \textit{ for all } w \in \mathcal{V}\}, \tag{3.2}$$

$$\mathcal{V}_c = \{v \in \mathcal{V} : a_{\mathcal{E}}(w,v) = 0 \textit{ for all } w \in \mathcal{V}_d\}, \tag{3.3}$$

with bilinear forms defined in (2.8) and (2.9). Furthermore, for $p \geq 2$ the following norm equivalence holds

$$c_1|||v|||^2 \leq |||v_c|||_{\mathcal{K}}^2 + |||v_d|||_{\mathcal{K}}^2 \leq c_2|||v|||^2, \tag{3.4}$$

with constants c_1 and c_2 independent of h but dependent on p.

For the proof of Theorem 3.1 we need the following two lemmas.

Lemma 3.1 *For each edge $E \in \mathcal{E}_I \cup \mathcal{E}_D$ there is a function $\varphi_E \in \mathcal{V}_d$ such that*

$$[\varphi_E] = 1 \quad \textit{on } E, \tag{3.5}$$

$$\int_{E'} [\varphi_E]v = 0 \quad \textit{for all } v \in \mathcal{P}_{p-1}(E') \textit{ and } E' \in \mathcal{E} \setminus E, \tag{3.6}$$

where $\mathcal{P}_{p-1}(E')$ denotes the space of polynomials of order $p-1$ defined on E'.

Proof. We consider the case $E \in \mathcal{E}_I$, the case $E \in \mathcal{E}_D$ is similar, and it is also easy to see that the proof does not work out for $E \in \mathcal{E}_N$. We construct φ_E elementwise. Let $K^+, K^- \in \mathcal{K}$ be the triangles which share an interior edge E. Let z denote the coordinate orthogonal to E and $H^\pm$ be the height of $K^\pm$, and let

$$\varphi_E|_K^\pm = L_p(2(z/H^\pm) \mp 1)/2, \tag{3.7}$$

where L_p denotes the Legendre polynomial, see [1], of order p defined on $[-1, 1]$. We begin by verifying that $\varphi_E \in \mathcal{V}_d$. Note that the condition

$$a_\mathcal{K}(w, v) - a_\mathcal{E}(w, v) = 0, \tag{3.8}$$

for all $w \in \mathcal{V}$, is equivalent to

$$-(\nabla \cdot \sigma(w), v)_K = (\sigma_n(w), \langle v \rangle)_{\partial K}, \tag{3.9}$$

for all $w \in \mathcal{V}_K$ and $K \in \mathcal{K}$. Note that it follows from the fact that the Legendre polynomial L_p is orthogonal to all polynomials of order $p-1$ that φ_E satisfies

$$-(\nabla \cdot \sigma(w), \varphi_E)_K = 0, \tag{3.10}$$

$$(\sigma_n(w), \langle \varphi_E \rangle)_{\partial K} = 0, \tag{3.11}$$

where in the last equality we also used that $\langle \varphi_E \rangle = 0$ on E. Thus φ_E is in $\mathcal{V}_d$. The properties (3.5) and (3.6) of φ_E are direct consequences of the construction. □

Lemma 3.2 *For $p \geq 2$ there is a $w \in \mathcal{V}_d$ for each $v \in \mathcal{V}$ such that*

$$\|h^{-1} P_0[v]\|_\mathcal{E}^2 = \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w) \rangle, P_0[v])_E, \tag{3.12}$$

$$|||w|||_\mathcal{K} \leq c \|h^{-1} P_0[v]\|_\mathcal{E}, \tag{3.13}$$

with constant c independent of h and p, and P_0 the edgewise L^2-projection on constant functions.

Proof. Let K be a triangle, E one of the edges of K, H the height of K orthogonal to E, and $z \in [0, H]$ the coordinate orthogonal to E. Then the normal derivative of the function $z(z/H - 1)$ is one on E and has average zero on the two other edges. Based on this observation and the fact that A is positive definite we conclude that for $p \geq 2$ we can construct a $w' \in \mathcal{V}_d$ for each $v \in \mathcal{V}$ such that

$$\sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w') \rangle, P_0[v])_E = \|h^{-1} P_0[v]\|_\mathcal{E}^2, \tag{3.14}$$

$$|||w'|||_\mathcal{K} \leq c \|h^{-1} P_0[v]\|_\mathcal{E}. \tag{3.15}$$

Next, for $p \geq 2$, we define $w \in \mathcal{V}_d$ by

$$a_{\mathcal{K}}(w, v) = a_{\mathcal{K}}(w', v) \quad \text{for all } v \in \mathcal{V}_d. \tag{3.16}$$

We note that setting $v = w$ and using the Cauchy-Schwarz inequality gives

$$|||w|||_{\mathcal{K}} \leq |||w'|||_{\mathcal{K}}, \tag{3.17}$$

and thus it follows that

$$|||w|||_{\mathcal{K}} \leq c\|h^{-1}P_0[v]\|_{\mathcal{E}}. \tag{3.18}$$

Using the definition of $\mathcal{V}_d$ we get

$$a_{\mathcal{E}}(w, v) = a_{\mathcal{E}}(w', v) \quad \text{for all } v \in \mathcal{V}_d, \tag{3.19}$$

and choosing $v = \varphi_E$, see Lemma 3.1, we find that

$$P_0\langle\sigma_n(w)\rangle = P_0\langle\sigma_n(w')\rangle \quad \text{ on } E, \tag{3.20}$$

for each edge $E \in \mathcal{E}_I \cup \mathcal{E}_D$. □

Remark 3.1 The construction of w' is a consequence of the classical nonconforming quadratic Morley element [14]. The degrees of freedom of the Morley element is the nodal values and the values of the normal derivative at the midpoints of the edges.

Lemma 3.3 *It holds*

$$\|h^{-1}(I - P_0)[v]\|_{\mathcal{E}} \leq c|||v|||_{\mathcal{K}} \quad \textit{for all } v \in \mathcal{V}, \tag{3.21}$$

with constant c independent of h and p, and P_0 the edgewise L^2-projection on constant functions.

Proof. Note that we may subtract the projection of v onto piecewise constants $\pi_0 v$ as follows

$$\begin{aligned}
\|h^{-1}(I - P_0)[v]\|_{\mathcal{E}}^2 &= \|h^{-1}(I - P_0)[v - \pi_0 v]\|_{\mathcal{E}}^2 && (3.22)\\
&\leq c \sum_{K \in \mathcal{K}} h^{-1}\|v - \pi_0 v\|_K \Big(h^{-1}\|v - \pi_0 v\|_K + \|v - \pi_0 v\|_{1,K}\Big) && (3.23)\\
&\leq c|||v|||^2, && (3.24)
\end{aligned}$$

where we finally used the interpolation estimate (4.12) together with the fact that the H^1-seminorm can be estimated by the energynorm. □

Proof of Theorem 3.1 Clearly $\mathcal{V} = \mathcal{V}_c + \mathcal{V}_d$ by the definition. Assume that $v \in \mathcal{V}_c \cap \mathcal{V}_d$. Then we conclude that $a_{\mathcal{K}}(v, v) = 0$ and thus v is a piecewise constant function. It follows that $a_{\mathcal{E}}(w, v) = 0$ for all $w \in \mathcal{V}_d$, invoking Lemma 3.2 we find that $v = 0$. Therefore the sum is direct for $p \geq 2$.

Starting with the left inequality in (3.4) we first observe that, using the inverse inequality (2.20) and the triangle inequality, we have

$$\begin{aligned}|||v|||^2 &\leq c|||v|||_{\mathcal{K}}^2 + \|h^{-1}[v]\|_{\mathcal{E}}^2 \\ &\leq c\Big(|||v_c|||_{\mathcal{K}}^2 + |||v_d|||_{\mathcal{K}}^2\Big) + \|h^{-1}[v]\|_{\mathcal{E}}^2, \qquad (3.25)\end{aligned}$$

and thus we need to estimate $\|h^{-1}[v]\|_{\mathcal{E}}^2$. Using the triangle inequality we have

$$\|h^{-1}[v]\|_{\mathcal{E}} \leq \|h^{-1}(I - P_0)[v]\|_{\mathcal{E}} + \|h^{-1}P_0[v]\|_{\mathcal{E}}. \qquad (3.26)$$

For the first term on the right hand side in (3.26) we have, using Lemma 3.3,

$$\|h^{-1}(I - P_0)[v]\|_{\mathcal{E}} \leq c|||v|||_{\mathcal{K}} \leq c\Big(|||v_d|||_{\mathcal{K}} + |||v_c|||_{\mathcal{K}}\Big). \qquad (3.27)$$

Next for the second, invoking Lemma 3.2 gives

$$\begin{aligned}\|h^{-1}P_0[v]\|_{\mathcal{E}}^2 &= \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w) \rangle, P_0[v])_E \qquad (3.28)\\ &= \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w) \rangle, [v])_E - \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w) \rangle, (I - P_0)[v])_E. \qquad (3.29)\end{aligned}$$

For the first term on the right hand side in (3.27) we have the estimate

$$\begin{aligned}\sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w) \rangle, [v])_E &= \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w), [v_d])_E \qquad (3.30)\\ &= \sum_{K \in \mathcal{K}} (\sigma(w), \nabla v_d)_K \qquad (3.31)\\ &\leq |||w|||_{\mathcal{K}} |||v_d|||_{\mathcal{K}} \qquad (3.32)\\ &\leq c\|h^{-1}P_0[v]\|_{\mathcal{E}} |||v_d|||_{\mathcal{K}}, \qquad (3.33)\end{aligned}$$

where we used the fact that $w \in \mathcal{V}_d$ in (3.30); the definition of $\mathcal{V}_d$ in (3.31); the Cauchy-Schwarz inequality in (3.32); and finally the stability estimate (3.13) in (3.33). For the second term

$$\begin{aligned}\sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(w) \rangle, (I - P_0)[v])_E &\leq \|\langle \sigma_n(w) \rangle\|_{\mathcal{E}} \|(I - P_0)h^{-1}[v]\|_{\mathcal{E}} \qquad (3.34)\\ &\leq c|||w|||_{\mathcal{K}} |||v|||_{\mathcal{K}} \qquad (3.35)\\ &\leq c\|h^{-1}P_0[v]\|_{\mathcal{E}} |||v|||_{\mathcal{K}}, \qquad (3.36)\end{aligned}$$

where we used the Cauchy-Schwarz inequality in (3.34), the inverse inequality (2.20) and Lemma 3.3 in (3.35), and finally the stability estimate (3.13) in (3.42).

Starting from (3.29) and using the triangle inequality together with estimates (3.33) and (3.42), and finally dividing with $\|h^{-1}P_0[v]\|_{\mathcal{E}}$, give

$$\|h^{-1}P_0[v]\|_{\mathcal{E}} \leq c\Big(|||v_d|||_{\mathcal{K}} + |||v_c|||_{\mathcal{K}}\Big), \tag{3.37}$$

which together with (3.26) and (3.27) prove the left inequality in (3.4).

We now turn to the proof of right inequality in (3.4). Starting from the definition (3.2) of $\mathcal{V}_d$ in Theorem 3.1, and setting $w = v_d$ we get

$$\begin{aligned} |||v_d|||_{\mathcal{K}}^2 &= a_{\mathcal{K}}(v_d, v_d) && (3.38)\\ &= a_{\mathcal{E}}(v_d, v) && (3.39)\\ &\leq \|\langle\sigma_n(v_d)\rangle\|_{\mathcal{E}}\|h^{-1}[v]\|_{\mathcal{E}} && (3.40)\\ &\leq c|||v_d|||_{\mathcal{K}}|||v|||, && (3.41) \end{aligned}$$

where we used the Cauchy-Schwarz inequality, and at last, the inverse inequality (2.20) and the obvious fact that $\|h^{-1}[v]\|_{\mathcal{E}} \leq |||v|||$. Finally, dividing by $|||v_d|||_{\mathcal{K}}$, and squaring both sides, give

$$|||v_d|||_{\mathcal{K}}^2 \leq c|||v|||^2. \tag{3.42}$$

Next for v_c we simply have

$$\begin{aligned} |||v_c|||_{\mathcal{K}}^2 &= |||v - v_d|||_{\mathcal{K}}^2 && (3.43)\\ &\leq c\Big(|||v|||_{\mathcal{K}}^2 + |||v_d|||_{\mathcal{K}}^2\Big) && (3.44)\\ &\leq c|||v|||^2, && (3.45) \end{aligned}$$

which together with (3.42) prove the right inequality in (3.4). At last tracing constants we find that both c_1^{-1} and c_2 are of the form $cC^2 + c$, where c denote constants independent of both h and p, and C is the constant in the inverse inequality (2.20), which depends on p. □

3.2 A two-scale formulation of the dG method

Here we shall derive a system of equations corresponding to (2.6) using the splitting given in Theorem 3.1. Writing $u = u_c + u_d$ and $v = v_c + v_d$ and using the following identities

$$\begin{aligned} a(u_c, v_d) &= 0, && (3.46)\\ a(u_d, v_c) &= 2a_{\mathcal{K}}(u_d, v_c), && (3.47)\\ a(u_d, v_d) &= a_{\mathcal{K}}(u_d, v_d), && (3.48) \end{aligned}$$

which are direct consequences of Theorem 3.1, we obtain a triangular system of the form: find $u = u_c + u_d \in \mathcal{V}_c + \mathcal{V}_d$ such that

$$\begin{aligned} a(u_c, v_c) + 2a_{\mathcal{K}}(u_d, v_c) &= l(v_c), \\ a_{\mathcal{K}}(u_d, v_d) &= l(v_d). \end{aligned} \tag{3.49}$$

We note that with this particular splitting of $\mathcal{V}$ the discontinuous scales, $\mathcal{V}_d$, are in fact not coupled to the continuous scales, $\mathcal{V}_c$.

3.3 Checkerboard solutions for $p = 1$

For $p = 1$ the splitting (3.1) in Theorem 3.1 is not direct and the norm equivalence (3.4) does not hold in general. This fact can be seen as follows. Using Green's formula we have

$$\begin{aligned} \sum_{K \in \mathcal{K}} (\nabla w, A\nabla v)_K = \sum_{K \in \mathcal{K}} (-\nabla \cdot \nabla w, v)_K \\ + \sum_{E \in \mathcal{E}_I} ([\sigma_n(w)], \langle v \rangle)_E + (\langle \sigma_n(w) \rangle, [v])_E + \sum_{E \in \mathcal{E}_D \cup \mathcal{E}_N} (\sigma_n(w), v)_E. \end{aligned}$$

Now if v is a piecewise constant function then $\nabla v = 0$, and if w is a piecewise linear function then $-\nabla \cdot A\nabla w = 0$ (recall that A is piecewise constant). Using these facts we get

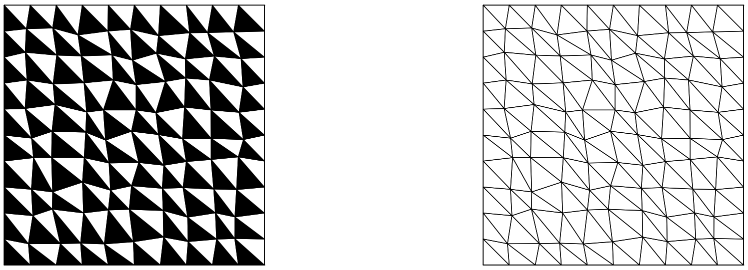
$$a_{\mathcal{E}}(w, v) = -\sum_{E \in \mathcal{E}_I} ([\sigma_n(w)], \langle v \rangle)_E - \sum_{E \in \mathcal{E}_N} (\sigma_n(w), v)_E,$$

and thus if $\mathcal{E}_N$ is empty and $\langle v \rangle = 0$ on each edge then $a_{\mathcal{E}}(w, v) = 0$ for all $w \in \mathcal{V}$. Going back to the splitting $\mathcal{V} = \mathcal{V}_c + \mathcal{V}_d$, in Theorem 3.1 we find that $v \in \mathcal{V}_c \cap \mathcal{V}_d$ and thus the splitting is not direct. Further it is easy to see that $|||v|||_{\mathcal{K}} = 0$, while $|||v|||^2 \neq 0$ and thus c_1 must be zero, i.e., (3.4) does not hold. However, a piecewise constant function v, with $\langle v \rangle = 0$ on each $E \in \mathcal{E}_I$, does only exist on a checkerboard mesh, i.e., a mesh which could be colored as a checkerboard with two colors. In Figure 2 we give an exampel of such a function v on an unstructured checkerboard triangulation of the unit square. In the case when $\mathcal{E}_N$ is not empty but the mesh is a checkerboard mesh we instead get that $c_1 \to 0$ as $h \to 0$. However, a general unstructured triangulation, is usually quite far from being a checkerboard mesh and in such a situation the norm equivalence will in general hold even for $p = 1$. See the computations of the inf-sup constant presented below.

4 Stability analysis and error estimates in the energy norm

4.1 Stability analysis

Our main result in this section is a proof that the inf-sup constant, see for instance [8], is positive independent of the meshsize. This stability result is, as is well known, key for



(a) (b)

Figure 2: (a) Checkerboard solution with black = -1 and white = 1 and (b) the corresponding triangulation of the unit square

proving existence and uniqueness of the discrete solution as well as error estimates in the energy norm.

Theorem 4.1 *If $p \geq 2$ then there is a constant $m > 0$, such that*

$$\inf_{u \in \mathcal{V}} \sup_{v \in \mathcal{V}} \frac{a(u,v)}{|||u|||\,|||v|||} \geq m. \tag{4.1}$$

The constant m is independent of h but depends on p.

Proof. Using identities (3.46–3.48) we have

$$a(u_c + u_d, v_c + v_d) = a(u_c, v_c) + 2a(u_c, v_d) + a(u_d, v_d). \tag{4.2}$$

Setting

$$v_c + v_d = u_c + \gamma u_d, \tag{4.3}$$

where $\gamma \in \mathbf{R}$ is a parameter, we get

$$\begin{aligned} a(u_c + u_d, v_c + v_d) &= |||u_c|||_{\mathcal{K}}^2 + 2a_{\mathcal{K}}(u_d, u_c) + \gamma |||u_d|||_{\mathcal{K}}^2 && (4.4) \\ &\geq |||u_c|||_{\mathcal{K}}^2 - 2\,|||u_d|||_{\mathcal{K}} |||u_c|||_{\mathcal{K}} + \gamma |||u_d|||_{\mathcal{K}}^2 && (4.5) \\ &\geq (1-\epsilon)|||u_c|||_{\mathcal{K}}^2 + (\gamma - \epsilon^{-1})|||u_d|||_{\mathcal{K}}^2. && (4.6) \end{aligned}$$

Here we used the Cauchy-Schwarz inequality and (2.21). Choosing ϵ such that $1 - \epsilon \geq m'$ and $\gamma \geq 1$ such that $\gamma - \epsilon^{-1} \geq m'$, we get

$$a(u_c + u_d, u_c + \gamma u_d) \geq m' \Big(|||u_c|||_{\mathcal{K}}^2 + |||u_d|||_{\mathcal{K}}^2 \Big). \tag{4.7}$$

Next we note that, for $\gamma \geq 1$, we have

$$c_1|||u_c + \gamma u_d|||^2 \leq |||u_c|||^2_{\mathcal{K}} + \gamma^2|||u_d|||^2_{\mathcal{K}} \tag{4.8}$$

$$\leq \gamma^2\Big(|||u_c|||^2_{\mathcal{K}} + |||u_d|||^2_{\mathcal{K}}\Big), \tag{4.9}$$

and thus we conclude that

$$|||u_c + u_d|||\,|||u_c + \gamma u_d||| \leq c_1^{-1}\gamma\Big(|||u_c|||^2_{\mathcal{K}} + |||u_d|||^2_{\mathcal{K}}\Big). \tag{4.10}$$

Combining (4.7) and (4.10) we imediately get the desired inf-sup bound

$$\inf_{u\in\mathcal{V}}\sup_{v\in\mathcal{V}} \frac{a(u,v)}{|||u|||\,|||v|||} \geq \frac{c_1 m'}{\gamma} = m. \tag{4.11}$$

□

Example: Computation of the inf-sup constant We compute the inf-sup constant for the discrete Laplacian defined by (2.6) on the unit square $\Omega = [0,1]^2$ with homogenous Dirichlet conditions on Γ. The triangulations are quasiuniform unstructured with N elements. For details on such computations we refer to Oden et al. [16]. In Table 1 we present the inf-sup constant m for a variety of triangulations and $p = 1, \ldots, 4$. We note that the inf-sup constant is independent of the number of elements (or meshsize) and decreases with increasing $p \geq 2$, as expected. Note also, that for $p = 1$ the inf-sup constant is indeed strictly positive due to the fact that these computations are done on an unstructured grid in two spatial dimensions, which is typically not close to a checkerboard mesh.

N	$p=1$	$p=2$	$p=3$	$p=4$
72	0.054	0.116	0.071	0.047
290	0.022	0.115	0.068	0.044
1300	0.022	0.115	0.067	0.044
2604	0.021	0.116	0.070	–
5366	0.023	0.115	–	–

Table 1: The inf-sup constant m for different p and meshes with N elements.

4.2 Error estimates in the energy norm

We first recall that given $u \in H^s(K)$, there is $\pi_K u \in \mathcal{P}_p(K)$ such that the following estimate holds

$$\|u - \pi_K u\|_{r,K} \leq c p_K^{r-s} h_K^{\mu-r} |u|_{s,K}, \tag{4.12}$$

where $0 \leq r \leq s$, $\mu = \min(p+1, s)$ and c is a constant independent of h and p, see [5]. Further we let $\pi u \in \mathcal{V}$ be defined by $(\pi v)|_K = \pi_K(v|_K)$. Using (4.12) we get the following lemma.

Lemma 4.1 *Let $u \in H^s$ then*

$$|||u - \pi u||| \leq c \left(\sum_{K \in \mathcal{K}} p_K^{-(2s-3)} h_K^{2(\mu-1)} |u|_{s,K}^2 \right)^{1/2}, \tag{4.13}$$

where $\pi u \in \mathcal{V}$ is defined in Subsection 2.2.

Proof. With $\eta = u - \pi u$ we have

$$|||\eta|||^2 = \|\eta\|_{\mathcal{K}}^2 + \|\langle \sigma_n(\eta) \rangle\|_{\mathcal{E}}^2 + \|h^{-1}[\sigma_n(\eta)]\|_{\mathcal{E}}^2.$$

Using the boundedness of A, we get $\|\eta\|_{\mathcal{K}} \leq c\|\eta\|_1$. For the second term we invoke the trace inequality (2.18), elementwise to obtain

$$\begin{aligned} \|\langle \sigma_n(\eta) \rangle\|_{\mathcal{E}}^2 &\leq c \sum_{K \in \mathcal{K}} h \|\nabla \eta\|_K \Big(h^{-1} \|\nabla \eta\|_K + \|\nabla \eta\|_{1,K} \Big) \\ &\leq c \sum_{K \in \mathcal{K}} \|\eta\|_{1,K} \Big(\|\eta\|_{1,K} + h \|\eta\|_{2,K} \Big). \end{aligned}$$

For the third term we get in the same way

$$\|h^{-1}[\eta]\|_{\mathcal{E}}^2 \leq c \sum_{K \in \mathcal{K}} h^{-1} \|\eta\|_K \Big(h^{-1} \|\eta\|_K + \|\eta\|_{1,K} \Big).$$

Now (4.13) follows directly from the interpolation error estimate (4.12). □

Using the stability estimates and interpolation error estimate we obtain the following energy norm estimate.

Theorem 4.2 *The following energy norm error estimate holds*

$$|||u - u_h|||^2 \leq c(1 + m^{-1}) \left(\sum_{K \in \mathcal{K}} p_K^{-(2s-3)} h_K^{2(\mu-1)} |u|_{s,K}^2 \right)^{1/2},$$

where c denotes constants independent of h and p. The constant m, defined in Theorem 4.1, is independent of h but depends on p.

Proof. First we write

$$|||u - u_h||| = |||u - \pi u||| + |||\pi u - u_h|||, \tag{4.14}$$

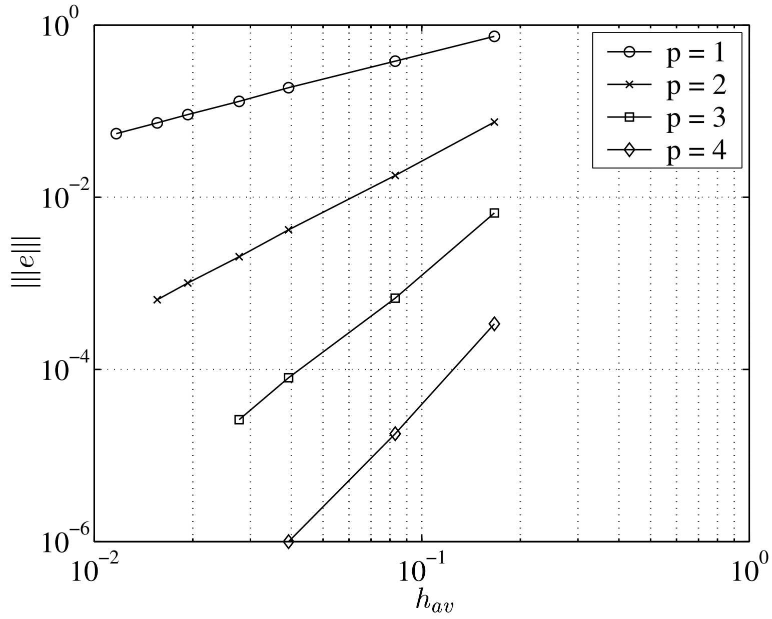


Figure 3: The energy error as a function of the average meshsize h_{av} for $p = 1, \ldots, 4$.

where π is the local L^2 projection on each element. Since $\pi u - u_h \in \mathcal{V}$ it follows from Theorem 4.1 that the second term can be estimated as follows

$$\begin{aligned} m|||\pi u - u_h||| &\leq \sup_{v \in V} \frac{a(\pi u - u_h, v)}{|||v|||} \\ &\leq |||u - \pi u|||, \end{aligned} \tag{4.15}$$

where in the second inequality we used Galerkin orthogonality (2.6) to get $a(\pi u - u_h, v) = a(\pi u - u, v)$, and then the Cauchy-Schwarz inequality gives $a(\pi u - u, v) \leq |||\pi u - u|||, |||v|||$. Together (4.14) and (4.15) give

$$|||u - u_h||| \leq (1 + m^{-1})|||u - \pi u|||,$$

and the right hand side can be now be immediately estimated using Lemma 4.1. $\square$

Example: The error in the energy norm. We consider the Poisson equation (2.1) on the unit square, $\Omega = [0, 1]^2$, with homogeneous Dirichlet boundary conditions, $u = 0$, on the boundary Γ and the right hand side f chosen so that the exact solution is $u(x, y) = \sin(\pi x)\sin(\pi y)$. The triangulation is unstructured and all triangles are of approximately the same size. We plot the error as a function of the average meshsize h_{av} defined by $h_{av} = 1/\sqrt{2N}$ where N is the number of elements.

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