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DYNAMIC SUBGRID MODELING FOR TIME DEPENDENT CONVECTION–DIFFUSION–REACTION EQUATIONS WITH FRACTAL SOLUTIONS

JOHAN HOFFMAN

Abstract. A dynamic scale similarity model is proposed. The subgrid model is tested for model problems related to time dependent non linear convection-diffusion-reaction systems with fractal solutions. The error of an approximate solution with subgrid model on a scale h is typically smaller than that of a solution without subgrid model on the scale $h/2$. We also consider the problem of a posteriori error estimation for fractal solutions, splitting the total computational error into a numerical error, related to the discretization of the continuous equations, and a modeling error, taking into account the quality of the subgrid model.

1. Introduction

In recent years methods of *dynamic subgrid modeling* have been proposed, in particular in turbulence modeling in *Dynamic Large Eddy Simulations* (DLES) by Germano *et.al.* [5]. The purpose of a subgrid model is to model the effect of unresolvable scales on resolvable scales corresponding to closure in turbulence modeling. The basic idea in dynamic subgrid modeling is to fit a particular subgrid model based on computed solutions on different resolvable scales, and then extrapolate the model to subgrid scales. In order for such a process based on scale extrapolation to work, it is necessary that the underlying problem has some 'scale regularity', so that the experience gained by fitting the model on a coarse scale with a fine scale solution as reference may be extrapolated to the finer scale. There is empirical evidence ([4, 13] and references therin) that many problems involving a range of scales from large to small, such as fluid flow at larger Reynolds numbers and flow in heterogeneous porous media, in fact do have such a regularity, once the larger scales related to the geometry of the particular problem have been resolved.

The purpose of this note is to study the feasibility of the indicated dynamic subgrid modeling procedure in the context of some model problems related to convection–diffusion–reaction systems with 'fractal' solutions, where the fractality originates from data. In this paper we study time dependent problems, extending the study of stationary problems in

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[6, 7, 8]. The relevance of studying fractal models is motivated by the abundant number of experimental observations of fractality in *e.g.* turbulent flows (see *e.g.* [13]).

2. Problem formulation

We consider the model problem: Find $u : \Omega \times [0,T] \to \mathbb{R}^n$ such that

$$\dot{u} + Lu = \dot{u} - \epsilon\Delta u + \beta \cdot \nabla u = f(u), \quad \forall (x,t) \in \Omega \times (0,T), \tag{2.1}$$

$$u = u_D, \quad \forall (x,t) \in \Gamma_D \times (0,T), \quad \frac{\partial u}{\partial n} = u_N, \quad \forall (x,t) \in \Gamma_N \times (0,T), \tag{2.2}$$

$$u(x,0) = u_0(x), \quad \forall x \in \Omega, \tag{2.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, $\Omega \subset \mathbb{R}^d$ and $\partial\Omega = \Gamma_D \bigcup \Gamma_N$. Typically we will assume that ϵ is small and that the solution u to (2.1)–(2.3) contains a range of scales, from very small scales to large scales, induced by either the initial condition $u_0(x)$ or the differential operator L through β. Assuming we want to find an approximation of u on the scale h, representing the finest spatial computational scale, we define for each fixed t the spatial *running average* u^h of u on the scale h by

$$u^h(x,t) = \frac{1}{h^d} \int_{x_1-h/2}^{x_1+h/2} \dots \int_{x_d-h/2}^{x_d+h/2} u(y,t)\ dy_1 \dots dy_d, \tag{2.4}$$

where we note that this operator commutes with space and time differentiation. Applying this operator to (2.1)–(2.3) we find that the running average u^h satisfies the equation

$$\dot{u}^h + L_h u^h = \dot{u}^h + \beta^h \cdot \nabla u^h - \epsilon\Delta u^h = f(u^h) + F_h(u), \quad u^h(x,0) = u_0^h(x), \tag{2.5}$$

where L_h is a simplified operator on the scale h resulting from approximating β by β^h and the correction term $F_h(u) = (f(u))^h - f(u^h) + L_h u^h - (Lu)^h$ contains the influence of the unresolved on u^h. We consider a computational problem without subgrid model of the form

$$\dot{u}_h + L_h u_h = f(u_h), \quad u_h(x,0) = u_0^h(x), \tag{2.6}$$

and a corresponding problem with subgrid model of the form

$$\dot{\tilde{u}}_h + L_h \tilde{u}_h = f(\tilde{u}_h) + \tilde{F}_h(\tilde{u}_h), \quad \tilde{u}_h(x,0) = u_0^h(x), \tag{2.7}$$

where $\tilde{F}_h(\tilde{u}_h)$ should approximate $F_h(u)$. In this paper we will consider a subgrid model of the form $\tilde{F}_h(\tilde{u}_h) = g(F_h(\tilde{u}_h), F_{2h}(\tilde{u}_h), F_{4h}(\tilde{u}_h))$, where the function g is derived based on a scale regularity assumption on $F_h(u)$.

3. Analysis of $F_h(u)$ using the Haar MRA

In the rest of this paper we let $\Omega = [0,1]^2$ and for each $h = 2^{-i}$, with $i = 0, 1, \dots$, we define a corresponding regular quadratic mesh τ^h with elements corresponding to subdomains $\Omega_{i,k}$ with side length h. We denote the space of piecewise constant functions on τ^h by V_i, and the closure of the union of the V_j's is equal to $L_2(\Omega)$. The chain of closed subspaces $V_0 \subset V_1 \subset \dots \subset V_j \subset \dots$ is denoted a *Haar Multiresolution Analysis* (MRA) of $L_2(\Omega)$ [1]. Each V_j is spanned by the dilates and integer translates of one *scale function* $\Phi \in V_0$, that is, $V_j = \mathrm{span}\{\Phi_{j,k}(x) = 2^j\Phi(2^jx-k)\}$. The functions $\Phi_{j,k}$ form an L_2-orthonormal basis in V_j, and we denote the orthogonal complement of V_j in V_{j+1} by W_j, which is generated by another orthonormal basis (the *wavelets*) $\Psi_{j,k}(x) = 2^j\Psi(2^jx-k)$, where $\Psi \in W_0$ is called the *mother wavelet.* $W_j = W_j^1 \oplus W_j^2 \oplus W_j^3$, where the W_j^ν's represent differences in the horizontal, vertical and diagonal directions respectively. The space $L_2(\Omega)$ can now be represented as the direct sum $L_2(\Omega) = V_0 \oplus W_0^1 \oplus ... \oplus W_0^3 \oplus ... \oplus W_j^1 \oplus ... \oplus W_j^3 \oplus ...$, and each $f \in L_2(\Omega)$ has a unique decomposition $f = f_\Phi\Phi + \sum_{j,k} f_{j,k}^1\Psi_{j,k}^1 + ... + f_{j,k}^3\Psi_{j,k}^3 = f_\Phi + \sum_j f_j^1 + ... + f_j^3$, where the f_j^ν's represent the contributions on the different scales 2^{-j}. For the one dimensional Haar MRA in $L_2([0,1])$, the scale function is defined by $\varphi(x) = 1$ for $x \in [0,1]$ and 0 else, and the mother wavelet is defined by $\psi(x) = 1$ for $x \in (0,1/2)$, -1 for $x \in (1/2,1)$ and 0 else. In two dimensions the scale function and the wavelets are tensor products of the one dimensional scale function and wavelets. For the two dimensional Haar MRA in $L_2(\Omega)$ we have the scale function $\Phi(x_1,x_2) = \varphi(x_1)\varphi(x_2)$ and the wavelets $\Psi^1(x_1,x_2) = \varphi(x_1)\psi(x_2)$, $\Psi^2(x_1,x_2) = \psi(x_1)\varphi(x_2)$, $\Psi^3(x_1,x_2) = \psi(x_1)\psi(x_2)$.

Definition 1. *For $f \in L_2(\Omega)$, we define $[f]^h = f_\varphi + \sum_{j<i} f_{j,k}^1\Psi_{j,k}^1 + f_{j,k}^2\Psi_{j,k}^2 + f_{j,k}^3\Psi_{j,k}^3$.*

The linear mapping $L_2 \ni f \to [f]^h \in V_i$ can then be identified with the L_2-projection of f onto V_i, and we note that $[f]^h = \bar{f}^h$, where $\bar{f}^h$ is the piecewise constant function on τ^h that equals f^h in the midpoints of each element in τ^h. If we let $\bar{F}_h(u)$ denote the piecewise constant function on τ^h that equals $F_h(u)$ in the midpoints of the elements of τ^h, we have

$$\bar{F}_h(u) = [f(u)]^h - f([u]^h) - ([\beta\cdot\nabla u]^h - [\beta]^h\cdot[\nabla u]^h),$$

which for second order reaction terms $f(u)$ leads us to model covariances of the form

$$E_h(v,w) = [vw]^h - [v]^h[w]^h, \tag{3.1}$$

for given functions v and w. The following lemma from [7] shows that $E_h(v,w)$ equals a weighted sum of the Haar coefficients on each scale finer than and equal to h.

Lemma 2. $E_h(v,w)(x) = \displaystyle\sum_{\substack{j\,\geq\, i \\ l\,:\,x\,\in\,\Omega_{j,l}}} 2^{2j}\ (v_{j,l}^1w_{j,l}^1 + v_{j,l}^2w_{j,l}^2 + v_{j,l}^3w_{j,l}^3).$

4. Scale extrapolation using self-similarity

By a *fractal* or *self-similar* function we mean in this paper a function that has some sort of scale similarity, following one of several definitions of a fractal by Mandelbrot [10]: *A fractal is a shape made of parts similar to the whole in some way.*

In Large Eddy Simulations (LES) the objective of subgrid modeling is to model the divergence of the exact Reynolds stress tensor $\tau_{ij} = (u_iu_j)^h - u_i^hu_j^h$, where each component has the form of a covariance similar to (3.1). There is an agreement of the existence of an inertial range in the energy spectrum of a turbulent flow [4], and different types of scale similarity assumptions also on the Reynolds stresses have been used to motivate various subgrid models, for example in the dynamic procedure [5] the parameters in a particular model are determined by comparing different averages of the resolved Reynolds stresses. In scale similarity models [12] the assumption is that the exact Reynolds stresses are proportional to the resolved Reynolds stresses. In [6, 7, 8] we have used an assumption based on the existence of such a scale similarity with respect to a Haar MRA generated by the hierarchy of successively refined computational meshes. This assumption is investigated in [9] for a transition study of a Couette flow. The finest computational scale is $h = 2^{-6}$ and the sum of the Haar coefficients for the Reynolds stress tensor component $\tau_{11}$are plotted for three different scales in Fig.1. Component τ_{11} is chosen since the streamwise velocity dominates the Couette flow. The transition to turbulence has started even though it is not fully developed, and the plot shows 60 elements in the coarsest mesh corresponding to the scale $8h$, and we see that the decrease in the Haar coefficients is reasonable regular and we consider this as some evidence of scale similarity of the Reynolds stresses for this flow, in the form of a power law for the sum of Haar coefficients on each scale. Scale similarity with respect to a Haar MRA has also been observed in experimental aerothermal data [11]. We will base our subgrid model on an Ansatz of the form

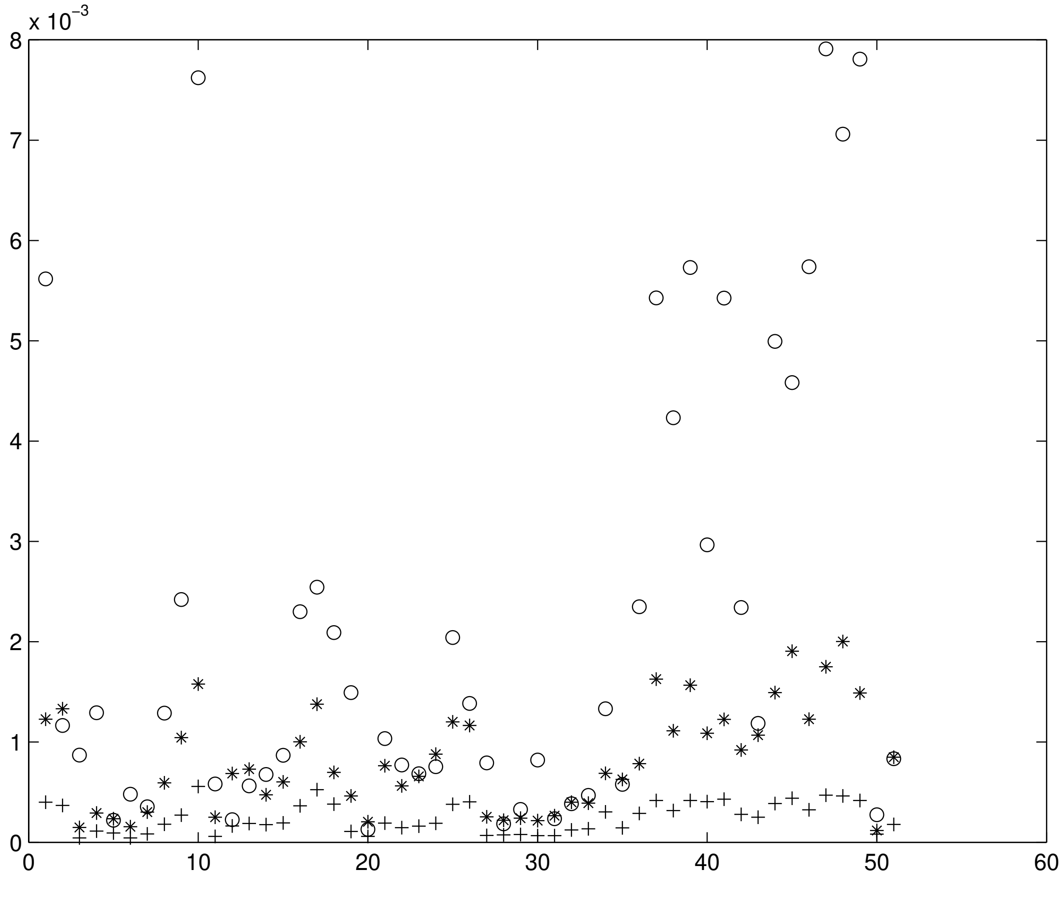

FIGURE 1. Sum of Haar coefficients for $\tau_{11}^{8h}(U_h)$ on scales $8h$ $('o')$, $4h$ $('*')$, and $2h$ $('+')$

$$E_h(v,w)(x) = C(x)h^{\mu(x)}, \quad x \in \Omega. \tag{4.1}$$

The Ansatz (4.1) can derived using Lemma 2 for the special case when the Haar coefficients of the functions v and w in (3.1) have the form $v^\nu_{j,k} = \alpha_\nu(x)2^{-j(1+\delta_\nu(x))}$ and $w^\nu_{j,k} = \beta_\nu(x)2^{-j(1+\gamma_\nu(x))}$ respectively, which corresponds to the simple self-similar forms $v_{j+1} = 2^{-\delta(x)}v_j$ and $w_{j+1} = 2^{-\gamma(x)}w_j$. We then have, assuming $\alpha_\nu, \beta_\nu, \delta_\nu, \gamma_\nu \in V_i$, that $E_h(v,w)(x) = C_1(x)h^{\mu_1(x)} + C_2(x)h^{\mu_2(x)} + C_3(x)h^{\mu_3(x)}$, where $C_\nu = \alpha_\nu\beta_\nu/(1-2^{-\mu_\nu})$ and $\mu_\nu = \delta_\nu + \gamma_\nu$. The proof is in [7], and for the case when $\delta_1 = \delta_2 = \delta_3$ and $\gamma_1 = \gamma_2 = \gamma_3$ we get (4.1). Lemma 2 further gives:

Proposition 3. $E_{2^k h}([v]^h, [w]^h) = E_{2^k h}(v,w) - E_h(v,w)$, $0 \le k \le i$.

Proof. By Lemma 2, $E_{2^k h}(v,w)$ is a sum over the scales $j \ge i-k$. Split this sum into two sums: one sum over the scales $j \ge i$ which is equal to $E_h(v,w)$ ($E_h(v,w) \in V_{i-k}$), and one sum over the scales $i-k \le j < i$ which is equal to $E_{2^k h}([v]^h, [w]^h)$. □

In the following we let $E_h(x)$ mean $E_h(v,w)(x)$ if nothing else is stated, and we now use the Ansatz (4.1) to derive the following propositions:

Proposition 4. *Assume that (4.1) is valid with $C, \mu \in V_{i-2}$, then*

$$E_h(v,w) = (E_{2h}(v,w) - E_h(v,w))/(\frac{E_{4h}(v,w) - E_{2h}(v,w)}{E_{2h}(v,w) - E_h(v,w)} - 1).$$

Proof. Using that $E_h, E_{2h} \in V_{i-2}$, (4.1) gives that $\dfrac{E_{4h} - E_{2h}}{E_{2h} - E_h} = \dfrac{C(4h)^\mu - C(2h)^\mu}{C(2h)^\mu - Ch^\mu} = 2^\mu$, and in the same way we have that $E_{2h} - E_h = Ch^\mu(2^\mu - 1) = E_h(2^\mu - 1) \Rightarrow E_h = (E_{2h} - E_h)/(2^\mu - 1)$ □

Proposition 5. *Assume that (4.1) is valid with $C, \mu \in V_{i-2}$, then*

$$E_h([v]^{2^{-n}h}, [w]^{2^{-n}h})(v,w) = (1 - (\frac{E_{4h}(v,w) - E_{2h}(v,w)}{E_{2h}(v,w) - E_h(v,w)})^{-n})E_h(v,w).$$

Proof. Proposition 3 $\Rightarrow E_h([v]^{2^{-n}h}, [w]^{2^{-n}h}) = E_h(v,w) - E_{2^{-n}h}(v,w) = Ch^\mu(1 - 2^{-n\mu})$ $= E_h(v,w)(1 - 2^{-n\mu})$, and we finally have 2^μ from the proof of Proposition 4. □

Proposition 3-5 motivates, for each t, an approximation of $[E_h(v,w)]^{4h}$, since an underlying assumption in Proposition 4-5 was that $E_h(v,w) \in V_{i-2}$. Each component of $\bar{F}_h(u)$ is a sum of terms of the form (3.1) and we use Proposition 4-5 to approximate each of these terms. We can use Proposition 3 to express these approximations in terms of $[v]^h, [w]^h$ instead of v, w, which is desirable since we will only have access to approximative solutions $\tilde{v}_h, \tilde{w}_h$ on the scale h. If $\tilde{F}_h(\tilde{u})_k$ denotes the approximation of $\bar{F}_h(u)_k$, the k:th component of $\bar{F}_h(u)$, and $2^{-n}h$ is the finest scale in the exact solution u, we have a subgrid model of the form

$$\bar{F}_h(u)_k = \sum E_h(v,w) \;\Rightarrow\; \tilde{F}_h(\tilde{u}_h)_k = \sum \tilde{E}_h(\tilde{v}_h, \tilde{w}_h), \tag{4.2}$$

$$\tilde{E}_h(\tilde{v}_h,\tilde{w}_h) = (1-(\frac{E_{4h}(\tilde{v}_h,\tilde{w}_h)-[E_{2h}(\tilde{v}_h,\tilde{w}_h)]^{4h}}{[E_{2h}(\tilde{v}_h,\tilde{w}_h)]^{4h}-[E_h(\tilde{v}_h,\tilde{w}_h)]^{4h}})^{-n})\frac{[E_{2h}(\tilde{v}_h,\tilde{w}_h)]^{4h}-[E_h(\tilde{v}_h,\tilde{w}_h)]^{4h}}{\frac{E_{4h}(\tilde{v}_h,\tilde{w}_h)-[E_{2h}(\tilde{v}_h,\tilde{w}_h)]^{4h}}{[E_{2h}(\tilde{v}_h,\tilde{w}_h)]^{4h}-[E_h(\tilde{v}_h,\tilde{w}_h)]^{4h}}-1}.$$

5. A posteriori error analysis

A general strategy for *a posteriori* error analysis based on duality is described in [3]. In this section we will make some remarks concerning the nonlinear problems with fractal solutions that appear in this paper. We assume that u is the solution to (2.1)-(2.3) and that U is a numerical approximation to the solution of either (2.6) or (2.7). The standard teqnique to obtain a posteriori error estimates for the error $e = u - U$ in a chosen norm is to introduce a linearized dual problem of the form: $-\dot{\varphi} - A(u,U)^*\varphi = 0,\ \varphi(T) = \varphi_T$, where $*$ denotes the transpose. The choice of φ_T determines in what norm we estimate e, for example, $\varphi_T = e(T)/\|e(T)\|$ gives a L_2-estimate of e. Here $\|\cdot\|$ denotes the L_2-norm and we further let $(\cdot,\cdot)$ denote the L_2-innner product. Following [3] (assuming continuous approximation in time) we get for $\varphi_T = e(T)/\|e(T)\|$ that

$$\|e(T)\| = (e(T), e(T)/\|e(T)\|) = (e(T),\varphi(T)) + \int_0^T (e, -\dot{\varphi} - A(u,U)^*\varphi)\ dt$$
$$= (e(0),\varphi(0)) + \int_0^T (\dot{e} + A(u,U)e,\varphi)\ dt.$$

Now if we choose $A(u,U)$ such that $A(u,U)e = Lu - f(u) - (LU - f(U))$ we get that

$$\|e(T)\| = (e(0),\varphi(0)) + \int_0^T (f(U) - \dot{U} - LU,\varphi)\ dt.$$

If $U = U_h$ is a numerical approximation of (2.6) we cannot expect the residual $f(U)-\dot{U}-LU$ to be small since (2.6) involves the simplified operator L_h and not L. If we on the other hand linearize at u^h instead of u and β^h instead of β, that is, we let $A(u,U)e = L_h u^h - f(u^h) - (L_h U - f(U))$, then $e(0) = 0$ (by a suitable choice of $U_h(0)$) and by (2.5) we get that

$$\|(u^h - U_h)(T)\| = \int_0^T (f(U_h) - \dot{U}_h - L_h U_h,\varphi)\ dt + \int_0^T (F_h(u),\varphi)\ dt, \tag{5.1}$$

where we now have a numerical residual, $R(U_h) = f(U_h) - \dot{U}_h - L_h U_h$, corresponding to the discretization error in approximating (2.6), and $F_h(u)$ which corresponds to a modeling residual, independent of U_h. If we on the other hand let $U = \tilde{U}_h$ be a numerical approximation of (2.7), we get that

$$\|(u^h - \tilde{U}_h)(T)\| = \int_0^T (\tilde{F}_h(\tilde{U}_h) + f(\tilde{U}_h) - \dot{\tilde{U}}_h - L_h\tilde{U}_h,\varphi)\ dt + \int_0^T (F_h(u) - \tilde{F}_h(\tilde{U}_h),\varphi)\ dt,$$

where we now have a numerical residual, $R(\tilde{U}_h) = \tilde{F}_h(\tilde{U}_h) + f(\tilde{U}_h) - \dot{\tilde{U}}_h - L_h\tilde{U}_h$, corresponding to the discretization error in approximating (2.7). The second term contains the difference $F_h(u) - \tilde{F}_h(\tilde{U}_h)$, which corresponds to a modeling residual one seeks to minimize by subgrid modeling.

We conclude that if u contains unresolvable subgrid scales, then the error with respect to u^h is the appropriate error to study, and thus the dual problem to be solved should be linearized at u^h and not at u. The dual problem then does not contain any scales finer than h, which means that it will be easy to solve and in particular no subgrid modeling is needed. This linearization also naturally splits the a posteriori error estimates in two terms, one involving a numerical residual and one involving a modeling residual, which enables us to balance the errors from discretization and subgrid modeling. In estimating the error $u^h - U_h$ by (5.1), $F_h(u)$ is unavaliable but can be approximated by the subgrid model $\tilde{F}_h(U_h)$. To estimate the difference $F_h(u) - \tilde{F}_h(\tilde{U}_h)$ further extrapolation seems to be needed.

6. Applications

We will now test the model (4.2) for some simple model problems in two space dimensions, where the scale similarity is introduced through data. We construct two dimensional fractal data as sums of local tensor products of the one dimensional fractal *Weierstrass function* $W_{\gamma,\delta}(x) = \gamma\ \sum_{j=0}^{N}\ 2^{-j\delta}\sin(2^j \cdot 2\pi x)$, where we let $\gamma = \delta = 0.1$ in all examples. The motivation for the test problems is that they contain important features of reactive flow problems. The first test problem is a recation-diffusion problem with a Volterra-Lotka type reaction term where the scale similarity appears in the initial data, simulating for example a reactor with two reacting spieces. We then add a small convection corresponding to a stirred reactor. The second test problem is a convection-diffusion problem where the scale similarity appears as the convecting velocity field, simulating a passive scalar in a turbulent velocity field. In all examples we will have $h = 2^{-5}$.

6.1. **Volterra-Lotka (VL).** We consider a reaction dominated problem of the form

$$\dot{u}_1 - \epsilon\Delta u_1 = u_1(1-u_2), \quad \dot{u}_2 - \epsilon\Delta u_2 + \beta\cdot\nabla u_2 = u_2(u_1 - 1),$$
$$\frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad u(x,0) = (W^{2D}_{\gamma,\delta}(x), 1),$$

where $\epsilon = 10^{-6}$, which corresponds to the classical Volterra-Lotka system with small diffusion and convection in one component. We have $F_h(u) = (-(u_1u_2)^h + u_1^h u_2^h, (u_1u_2)^h - u_1^h u_2^h)$, and by (4.2) we have that $\tilde{F}_h(\tilde{u}_h) = (-\tilde{E}_h(\tilde{u}_1, \tilde{u}_2), \tilde{E}_h(\tilde{u}_1, \tilde{u}_2))$, with $n = 4$ (reference scale minus computational scale). For these problems we use a central difference-Crank-Nicolson scheme for the midpoints of the elements, where we regard these midpoint values to represent a piecewise constant approximation over the elements, and the reference scale is 2^{-9} in the computation of the error. The solutions are oscillating and both u_1 and u_2 are fractal for $t > 0$, even though $u_2(x,0) = 1$. We want to approximate u^h, and the errors $\|u^h - U\|$ are shown in Tab.1-3, where U is the error without model, without model but computed on the finer scale $h/2$ and then projected onto the scale h, with the model (4.2), and with the model $\tilde{F}_h(\tilde{u}_h) = F_{2h}(\tilde{u}_h, \tilde{u}_h)$ corresponding to $n = 1$, meaning that we assume that no finer scales than $h/2$ are present in the exact solution u. We

TABLE 1. Error in $u_1$in L_1-norm for VL with $\beta = 0$

subgrid model	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
no model	$1.9{\cdot}10^{-3}$	$7.2{\cdot}10^{-3}$	$1.1{\cdot}10^{-2}$	$1.1{\cdot}10^{-2}$
no model on $h/2$	$1.4{\cdot}10^{-3}$	$4.6{\cdot}10^{-3}$	$7.0{\cdot}10^{-3}$	$6.8{\cdot}10^{-3}$
$\tilde{F}_h(\tilde{u}_h) = F_{2h}(\tilde{u}_h, \tilde{u}_h)$	$1.4{\cdot}10^{-3}$	$4.8{\cdot}10^{-3}$	$6.5{\cdot}10^{-3}$	$6.3{\cdot}10^{-3}$
model (4.2)	$1.0{\cdot}10^{-3}$	$2.8{\cdot}10^{-3}$	$3.8{\cdot}10^{-3}$	$4.2{\cdot}10^{-3}$

TABLE 2. Error in $u_2$in L_1-norm for VL with $\beta = 0$

subgrid model	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
no model	$1.4{\cdot}10^{-4}$	$3.4{\cdot}10^{-4}$	$2.0{\cdot}10^{-4}$	$4.4{\cdot}10^{-4}$
no model on $h/2$	$1.0{\cdot}10^{-4}$	$2.1{\cdot}10^{-4}$	$1.2{\cdot}10^{-4}$	$2.9{\cdot}10^{-4}$
$\tilde{F}_h(\tilde{u}_h) = F_{2h}(\tilde{u}_h, \tilde{u}_h)$	$9.0{\cdot}10^{-5}$	$2.0{\cdot}10^{-4}$	$1.5{\cdot}10^{-4}$	$2.8{\cdot}10^{-4}$
model (4.2)	$7.0{\cdot}10^{-5}$	$1.4{\cdot}10^{-4}$	$1.5{\cdot}10^{-4}$	$2.4{\cdot}10^{-4}$

TABLE 3. Error in $u_1$in L_1-norm for VL with $\beta \neq 0$

subgrid model	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
no model	$2.5{\cdot}10^{-5}$	$1.5{\cdot}10^{-4}$	$3.4{\cdot}10^{-4}$	$6.4{\cdot}10^{-4}$
no model on $h/2$	$2.5{\cdot}10^{-5}$	$1.1{\cdot}10^{-4}$	$1.9{\cdot}10^{-4}$	$3.3{\cdot}10^{-4}$
$\tilde{F}_h(\tilde{u}_h) = F_{2h}(\tilde{u}_h, \tilde{u}_h)$	$2.5{\cdot}10^{-5}$	$1.1{\cdot}10^{-4}$	$2.1{\cdot}10^{-4}$	$5.0{\cdot}10^{-4}$
model (4.2)	$2.0{\cdot}10^{-5}$	$6.0{\cdot}10^{-5}$	$2.0{\cdot}10^{-4}$	$5.6{\cdot}10^{-4}$

first let $\beta = 0$ and compute to $T = 2$, then we let β be a rotational mixing of order h: $\beta = h\ (\sin(\pi x_1)\cos(\pi x_2), -\cos(\pi x_1)\sin(\pi x_2))$ and here we only compute to $T = 1$, since after $T = 1$ the subgrid scales are dominated by the convective streaks due to β. We study the error for each component individually, and for $\beta = 0$ we find that the solution with the subgrid model (4.2) is the best for both components, even though the modeling errors are smaller in u_2 since $u_2(x, 0)$ is constant. We also note that the solution with the subgrid model corresponding to $n = 1$ is very close to the solution on the scale $h/2$ without sungrid model as expected. For $\beta \neq 0$ the solution with subgrid model (4.2) is best for u_1 but, because of the convection, subgrid scales in u_2 do not develop and the solutions with subgrid models does not differ significally from the solution without subgrid model. The solution on $h/2$ is better since the numerical error is then reduced.

6.2. **Fractal convection (FC).** We also consider a convection dominated problem of the form

$$\dot{u} + \beta \cdot \nabla u - \epsilon \Delta u = 1, \quad u|_{x_1=0,x_2=0} = 0, \quad \frac{\partial u}{\partial n}|_{x_1=1,x_2=1} = 0, \quad u(x, 0) = 0, \tag{6.1}$$

for $\beta = (W^{2D}_{\gamma,\delta}, W^{2D}_{\gamma,\delta})$ and $\epsilon = 10^{-3}$, which we solve by a Streamline Diffusion cG(1)cG(1) method [2] with bilinear elements. In the computation of the error the reference scale is 2^{-8}. The solution is in this case relatively smooth since the fractal β is only acting on

TABLE 4. Error in L_1-norm for FC

subgrid model	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1.0$
no model	$7.2{\cdot}10^{-3}$	$1.2{\cdot}10^{-2}$	$1.1{\cdot}10^{-2}$	$1.1{\cdot}10^{-2}$
model (4.2)	$5.8{\cdot}10^{-3}$	$7.2{\cdot}10^{-3}$	$6.2{\cdot}10^{-3}$	$6.2{\cdot}10^{-3}$

the derivatives of the solution. We have $F_h(u) = \beta^h \cdot (\nabla u)^h - (\beta \cdot \nabla u)^h$, and in Tab.4 we see that the error in the solution with the subgrid model (4.2) is less than in the solution without subgrid model.

7. Conclusions

A dynamic scale similarity model for convection-diffusion-reaction problems with fractal solutions was proposed, based on a scale regularity Ansatz with respect to a Haar MRA generated by the hierarchy of successively refined computational meshes. In computational experiments the solution to a problem on the scale h with subgrid model was better than the solution to a problem on scale $h/2$ without subgrid model. We also considered the problem of a posteriori error estimation for these problems. We found that by linearize the dual problem at u^h, and not at u, the error representation naturally split into separate terms involving a numerical residual and a modeling residual respectively, which enables us to balance errors from dicretization and subgrid modeling.

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