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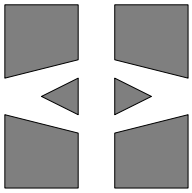

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NITSCHE'S METHOD FOR COUPLING NON-MATCHING MESHES IN FLUID-STRUCTURE VIBRATION PROBLEMS

PETER HANSBO AND JOAKIM HERMANSSON

Abstract. Nitsche's method [9] is a classical method for imposing essential boundary conditions weakly. Unlike the penalty method, it is consistent with the original differential equation. The strong point of Nitsche's method is that it retains the convergence rate of the underlying finite element method, whereas the standard penalty method either requires a "very large" penalty parameter, destroying the condition number of the resulting matrix problem, or, in case the condition number is to be retained, is limited to first order energy-norm accuracy. In this paper, we give a formulation of Nitsche's method suitable for the problem of fluid-structure interaction. Numerical examples are given.

1. Introduction

Nitsche's method [9], which was introduced in the early seventies, is a method for imposing essential boundary conditions weakly in the finite element method (FEM) approximation of elliptic problems. Basically, Nitsche's method consists of imposing the essential (Dirichlet) boundary conditions via three boundary terms: two containing weak forms of the normal derivatives of the solution and the test functions, and one containing a mesh–dependent term penalizing the deviation from the correct boundary condition. The normal derivatives are added so as to make the method consistent and symmetric, and the penalty term is added to make the method stable. Thus, the penalty term does not play the same role as in a pure penalty method: in a standard penalty method there is a consistency error depending on the size of the penalty parameter which typically is chosen "large" to minimize the effect of this error. In Nitsche's method, which is consistent, the penalty parameter can be kept at $O(h^{-1})$, where h is the element size, irrespective of the degree of the polynomial approximation.

In a recent paper, Becker, Hansbo, and Stenberg [4] extended Nitsche's method to the case of the coupling of non-matching grids for the finite element solution of Poisson's equation. Further developments of this approach were made in Hansbo & Hansbo [6], where arbitrarily cut meshes for the approximation of problems with discontinuous coefficiens were considered, in Hansbo, Hansbo & Larson [7], where overlapping meshes were used, and

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in Burman & Hansbo [3] where the coupling of Darcian and Stokesian flow was considered. In this paper, we further extend the concept to deal with multi–physics problems: the coupling of fluid and structure meshes in elastoacoustic vibration analysis.

2. The continuous problem

We shall consider the eigenproblem described by linear elasticity coupled with an irrotational fluid in $\mathbb{R}^d$, where $d = 2$ or $d = 3$. Given the the density of the fluid, ρ_{F}, and the solid, ρ_{S}, and the acoustic speed c, we seek the frequency ω, the velocity field $\boldsymbol{u} = (u_i)_{i=1}^d$, and the pressure p such that

$$\nabla p - \omega^2 \rho_{\mathrm{F}} \boldsymbol{u}_F = \boldsymbol{0} \quad \text{in } \Omega_{\mathrm{F}}, \tag{2.1}$$

$$p + c^2 \rho_{\mathrm{F}} \nabla \cdot \boldsymbol{u}_F = 0 \quad \text{in } \Omega_{\mathrm{F}}, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}_S) + \omega^2 \rho_{\mathrm{S}} \boldsymbol{u}_S = \boldsymbol{0} \quad \text{in } \Omega_{\mathrm{S}}, \tag{2.3}$$

$$\sigma_n(\boldsymbol{u}_S) + p = 0 \quad \text{on } \Gamma, \tag{2.4}$$

$$\boldsymbol{\sigma}_t(\boldsymbol{u}_S) = \boldsymbol{0} \quad \text{on } \Gamma, \tag{2.5}$$

$$(\boldsymbol{u}_F - \boldsymbol{u}_S) \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{2.6}$$

$$\boldsymbol{u}_S = \boldsymbol{0} \quad \text{on } \partial\Omega_{\mathrm{D}} \tag{2.7}$$

Here, the components of the stress tensor are given by

$$\sigma_{ij} = \lambda \delta_{ij} \nabla \cdot \boldsymbol{u} + 2\mu \, \varepsilon_{ij}(\boldsymbol{u}),$$

where δ_{ij} is the Kronecker delta,

$$\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

and λ and μ are the Lamé constants. Further, $\sigma_n = \boldsymbol{n} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})$ is the normal stress on Γ, and $\boldsymbol{\sigma}_t = \boldsymbol{\sigma} \cdot \boldsymbol{n} - \sigma_n \boldsymbol{n}$ is the tangential traction vector on Γ. We assume that the fluid domain is completely surrounded by the solid domain, and let $\boldsymbol{n}$ denote the outward pointing normal to Ω_F on the interface Γ that separates Ω_F from Ω_S.

To formulate the problem in weak form, we introduce the function space

$$V := \{(\boldsymbol{u}_F, \boldsymbol{u}_S) \in H(\mathrm{div}, \Omega_F) \times H^1(\Omega_S),\ [\![\boldsymbol{u} \cdot \boldsymbol{n}]\!] = 0 \text{ on } \Gamma, \\ \boldsymbol{u}_S = 0 \text{ on } \partial\Omega_D\},$$

where $[\![\boldsymbol{u} \cdot \boldsymbol{n}]\!] := (\boldsymbol{u}_F - \boldsymbol{u}_S) \cdot \boldsymbol{n}$ and

$$H(\mathrm{div}, \Omega_F) = \{\boldsymbol{v} \in L_2(\Omega_F) : \|\nabla \cdot \boldsymbol{v}\|_{L_2(\Omega_F)} < \infty\}.$$

The continuous problem can now be stated in weak form as follows: find $(\boldsymbol{u}_F, \boldsymbol{u}_S) \in V$ and $\omega^2 \in \mathbb{R}$ such that

$$\begin{aligned} (\rho_F c^2 \nabla \cdot \boldsymbol{u}_F, \nabla \cdot \boldsymbol{v}_F)_{\Omega_F} &+ (\boldsymbol{\sigma}(\boldsymbol{u}_S), \boldsymbol{\varepsilon}(\boldsymbol{v}_S))_{\Omega_S} \\ &= (\omega^2 \rho_F \boldsymbol{u}_F, \boldsymbol{v}_F)_{\Omega_F} + (\omega^2 \rho_S \boldsymbol{u}_S, \boldsymbol{v}_S)_{\Omega_S} \end{aligned} \tag{2.8}$$

for all $(\boldsymbol{v}_F, \boldsymbol{v}_S) \in V$. Here we have used the notation

$$(\boldsymbol{\sigma}(\boldsymbol{u}_S), \boldsymbol{\varepsilon}(\boldsymbol{v}_S))_{\Omega_S} := \int_{\Omega_S} \sum_{ij} \sigma_{ij}(\boldsymbol{u}_S)\varepsilon_{ij}(\boldsymbol{v}_S)\, dx,$$

$$(\boldsymbol{u}_S, \boldsymbol{v}_S)_{\Omega_S} := \int_{\Omega_S} \boldsymbol{u}_S \cdot \boldsymbol{v}_S\, dx,$$

and so on.

In (2.8), the zero jump condition is hidden in the definition of the function spaces, whereas in a finite element method (in particular with different kinds of approximations in the different domains) this typically has to be modeled explicitly. In the following Section, we will incorporate the zero jump condition weakly in a way that allows for arbitrary approximations on either side of the interface. We emphasize that this coupling is in no way related to the specific choice of discrete spaces we employ in this paper.

3. The finite element method

In this Section, we shall extend the finite element method for non-matching meshes introduced in [4] to the problem (2.1)–(2.7). To formulate our method, we suppose that we have regular finite element partitionings $\mathcal{T}_h^i$, $i = 1, 2$, of the subdomains $\Omega_1 := \Omega_F$ and $\Omega_2 := \Omega_S$ into shape regular simplexes K. These two meshes imply the existence of trace meshes on the interface

$$\mathcal{G}_h^i = \{\ E\ : E = K \cap \Gamma,\ K \in \mathcal{T}_h^i\ \}. \tag{3.1}$$

By h_K and h_E we denote the diameter of element $K \in \mathcal{T}_h^i$ and $E \in \mathcal{G}_h^i$, respectively. Next, we introduce the Raviart-Thomas finite element spaces

$$\mathcal{R}T_k := \{\boldsymbol{u} \in H(\mathrm{div}, \Omega_F) :\ \boldsymbol{u}|_K \in (P_k(K))^2 + \boldsymbol{x}P_k(K)\}$$

where $P_k(K)$ is the space of k :th degree polynomials on the element K (cf. Brezzi & Fortin [5]). Further, we let

$$W_l := \{\boldsymbol{u} \in H^1(\Omega_S) :\ \boldsymbol{u} = 0 \text{ on } \partial\Omega_D,\ \boldsymbol{u}|_K \in P_l(K)\}.$$

Remark 1. *The construction of the basis functions for the Raviart-Thomas spaces is perhaps most easily done in the physical configuration, as opposed to standard FEM. Consider an element in 2D with nodes $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_2$ and with side normals*

$$\begin{aligned}\boldsymbol{n}_1 &:= (y_2 - y_1, x_1 - x_2)/l_1,\\ \boldsymbol{n}_2 &:= (y_3 - y_2, x_2 - x_3)/l_2,\\ \boldsymbol{n}_3 &:= (y_1 - y_3, x_3 - x_1)/l_3,\end{aligned}$$

where the l_i denote the side lengths. The lowest order Raviart-Thomas approximation can be written [5]*:*

$$\boldsymbol{\varphi}_i = \begin{bmatrix} a_i \\ a_i \end{bmatrix} + \begin{bmatrix} b_i x \\ c_i y \end{bmatrix},$$

and to construct the vector-valued basis $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3\}$ on the element we need to solve

$$\boldsymbol{\varphi}_i \cdot \boldsymbol{n}_i = \delta_{ij} \quad \textit{at } \boldsymbol{x} = (\boldsymbol{x}_i + \boldsymbol{x}_{i+1})/2 \textit{ (modulo 3)}.$$

This is a simple linear algebra problem

$$\boldsymbol{S} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \boldsymbol{f}_i,$$

with $(\boldsymbol{f}_i)_j = \delta_{ij}$ *and*

$$\boldsymbol{S} := \begin{bmatrix} \dfrac{x_1-x_2}{l_1} + \dfrac{y_2-y_1}{l_1} & \dfrac{(x_1+x_2)\,(y_2-y_1)}{2\,l_1} & \dfrac{(x_1-x_2)\,(y_1+y_2)}{2\,l_1} \\ \dfrac{x_2-x_3}{l_2} + \dfrac{y_3-y_2}{l_2} & \dfrac{(x_2+x_3)\,(y_3-y_2)}{2\,l_2} & \dfrac{(x_2-x_3)\,(y_2+y_3)}{2\,l_2} \\ \dfrac{x_3-x_1}{l_3} + \dfrac{y_1-y_3}{l_3} & \dfrac{(x_1+x_3)\,(y_1-y_3)}{2\,l_3} & \dfrac{(x_3-x_1)\,(y_1+y_3)}{2\,l_3} \end{bmatrix}.$$

It is well known that restricting the approximation by using standard continuous finite elements in the fluid domain will give rise to spurious eigensolutions with nonzero eigenvalues interspersed among the "real" ones. The use of Raviart-Thomas elements, which are tailor-made for approximating vector fields where only normal continuity can be assumed, will alleviate this problem. In particular, Rodríguez & al. (e.g., [1, 2]) have shown that the approximation using the combination $\mathcal{R}T_0$ and W_1 is free from spurious solution and gives optimal order error estimates for the eigenvectors and eigenvalues. In their work, a particular projection scheme for the coupling on the interface was used (equivalent to the Lagrange multiplier method discussed in Section 3.1). Here, we instead consider the use of Nitsche's method at the interface, which in particular allows for arbitrary combinations of polynomial approximation on the different domains.

We now consider the problem of finding $(\boldsymbol{U}_F, \boldsymbol{U}_S) \in \mathcal{R}T_k \times W_l$ and $\lambda^h \in \mathbb{R}$ such that

$$a_h(\boldsymbol{U}, \boldsymbol{v}) = \lambda^h\, b(\boldsymbol{U}, \boldsymbol{v}) \quad \forall (\boldsymbol{v}_F, \boldsymbol{v}_S) \in \mathcal{R}T_k \times W_l, \tag{3.2}$$

where

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) \;:=\; & (\rho_F c^2 \nabla \cdot \boldsymbol{u}_F, \nabla \cdot \boldsymbol{v}_F)_{\Omega_F} \\ & + (\boldsymbol{\sigma}(\boldsymbol{u}_S), \boldsymbol{\varepsilon}(\boldsymbol{v}_S))_{\Omega_S} - (\{\sigma_n(\boldsymbol{u})\}, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_\Gamma \\ & - (\{\sigma_n(\boldsymbol{v})\}, [\![\boldsymbol{U} \cdot \boldsymbol{n}]\!])_\Gamma + \gamma \textstyle\sum_{E \in \mathcal{G}_h^i} h_E^{-1} ([\![\boldsymbol{U} \cdot \boldsymbol{n}]\!], [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_E, \end{aligned} \tag{3.3}$$

$$b(\boldsymbol{u}, \boldsymbol{v}) := (\rho_F \boldsymbol{u}_F, \boldsymbol{v}_F)_{\Omega_F} + (\rho_S \boldsymbol{u}_S, \boldsymbol{v}_S)_{\Omega_S}, \tag{3.4}$$

and λ^h is an approximation of ω^2. Here, we have used the notation

$$\{\sigma_n(\boldsymbol{u})\} := \alpha \sigma_n(\boldsymbol{u}_S) + (1-\alpha) \rho_F c^2 \nabla \cdot \boldsymbol{u}_F,$$

with $0 \le \alpha \le 1$, i.e., a convex combination of the (discrete) normal stresses on the surface Γ. The number α can be chosen freely between 0 and 1, e.g., from implementation considerations; indeed, it may even be chosen differently on different elements. We shall in the following choose $\alpha = 0$, i.e.,

$$\{\sigma_n(\boldsymbol{u})\} := \rho_F c^2 \nabla \cdot \boldsymbol{u}_F,$$

and we can then take the sum exclusively over the edges $E \in \mathcal{G}_h^1$ in (3.3). The parameter γ must be chosen from a perspective of stability and can neither be chosen too small or

too large. A lower bound for γ is easily computed, however (cf. below), and in order to have a well conditioned problem it is prudent not to choose γ much larger than this lower bound. In our experience, the computed solution is, in the current context, less sensitive to the choice of γ for the particular choice $\alpha = 0$. This choice also has a strong relation to the work of Rodríguez & al. [1, 2], which we will discuss below.

We note that a solution $(\boldsymbol{u}_F, \boldsymbol{u}_S, \omega^2)$ to (2.1)–(2.7) also satisfies (3.2). Taking the gradient of (2.2), subtracting (2.1), multiplying by $\boldsymbol{v}_F \in \mathcal{R}T_k$ and using Green's formula, we obtain

$$(\omega^2 \rho_F \boldsymbol{u}_F, \boldsymbol{v}_F)_{\Omega_F} = (\rho_F c^2 \nabla \cdot \boldsymbol{u}_F, \nabla \cdot \boldsymbol{v}_F)_{\Omega_F} - (\rho_F c^2 \nabla \cdot \boldsymbol{u}_F, \boldsymbol{v}_F \cdot \boldsymbol{n})_\Gamma.$$

Similarly, multiplying (2.3) by $\boldsymbol{v}_S \in W_l$ and using Green's formula we find

$$(\omega^2 \rho_S \boldsymbol{u}_S, \boldsymbol{v}_S)_{\Omega_S} = (\boldsymbol{\sigma}(\boldsymbol{u}_S), \boldsymbol{\varepsilon}(\boldsymbol{v}_S))_{\Omega_S} + (\boldsymbol{\sigma}(\boldsymbol{u}_S) \cdot \boldsymbol{n}, \boldsymbol{v}_S)_\Gamma.$$

Now, using that $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{\sigma}_t + \sigma_n \boldsymbol{n} = \sigma_n \boldsymbol{n}$ on Γ, together with (2.4) and (2.2) and the fact that, since $[\![\boldsymbol{u} \cdot \boldsymbol{n}]\!] = 0$ on Γ,

$$0 = -(\{\sigma_n(\boldsymbol{v})\}, [\![\boldsymbol{u} \cdot \boldsymbol{n}]\!])_\Gamma + \gamma \sum_{E \in \mathcal{G}_h^1} h_E^{-1} ([\![\boldsymbol{u} \cdot \boldsymbol{n}]\!], [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_E,$$

we find that $(\boldsymbol{u}_F, \boldsymbol{u}_S, \omega^2)$ satisfy (3.2). Thus the method is "arbitrary order consistent" and there is no conformity error that has to be dominated by a large penalty parameter as in a pure penalty method.

For the stability analysis below we need the following mesh-dependent "discrete half-norms":

$$\|\boldsymbol{v}\|_{1/2,h,\Gamma}^2 := \sum_{E \in \mathcal{G}_h^1} h_E^{-1} \|\boldsymbol{v}\|_{L_2(E)}^2 \tag{3.5}$$

and

$$\|\boldsymbol{v}\|_{-1/2,h,\Gamma}^2 := \sum_{E \in \mathcal{G}_h^1} h_E \|\boldsymbol{v}\|_{L_2(E)}^2, \tag{3.6}$$

which satisfy

$$|(\boldsymbol{v}, \boldsymbol{w})_\Gamma| \leq \|\boldsymbol{v}\|_{1/2,h,\Gamma} \|\boldsymbol{w}\|_{-1/2,h,\Gamma}. \tag{3.7}$$

We further define the energy-like norm

$$\|\boldsymbol{v}\|_{E,h}^2 = (\boldsymbol{\sigma}(\boldsymbol{v}_S), \boldsymbol{\varepsilon}(\boldsymbol{v}_S))_{\Omega_S} + \|\rho_F^{1/2} c \nabla \cdot \boldsymbol{v}_F\|_{L_2(\Omega_F)}^2 + \gamma \|[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]\|_{1/2,h,\Gamma}^2. \tag{3.8}$$

The following estimate is readily proven by scaling from a reference element, see [8].

Lemma 1. *There is a positive constant C_I, independent of h, ρ_F, and c, such that*

$$\left\| \rho_F^{1/2} c \, \nabla \cdot \boldsymbol{v}_F \right\|_{-1/2,h,\Gamma}^2 \leq C_I (\rho_F c^2 \nabla \cdot \boldsymbol{v}_F, \nabla \cdot \boldsymbol{v}_F)_{\Omega_F} \quad \forall \boldsymbol{v}_F \in \mathcal{R}T_k. \tag{3.9}$$

A bound for the constant C_I can be computed by solving a small eigenvalue problem, cf. Hansbo & Larson [8]. In the case of $\mathcal{R}T_0$, $\nabla \cdot \boldsymbol{v}_F$ is a constant and we directly find, on an element K with an edge E on Γ, that

$$\int_E h_E \, \rho_F \, c^2 (\nabla \cdot \boldsymbol{v}_F)^2 \, ds = h_E^2 \, \rho_F \, c^2 (\nabla \cdot \boldsymbol{v}_F)^2 = \frac{h_E^2}{|K|} \int_K \rho_F \, c^2 (\nabla \cdot \boldsymbol{v}_F)^2 \, dx,$$

where $|K|$ is the area of the element, so $C_I = \max_K h_E^2/|K|$ depends only on the shape of the elements (which are assumed to be shape regular).

It is now straightforward to show that the bilinear form $a_h(\cdot,\cdot)$ is positive on $\mathcal{R}T_k \times W_l$: by (3.7), we have

$$a_h(\boldsymbol{U},\boldsymbol{U}) \;\; \geq \;\; \|\boldsymbol{U}\|_{E,h}^2 - 2\|\rho_F^{1/2} \, c \, \nabla \cdot \boldsymbol{U}_F\|_{-1/2,h,\Gamma} \|\rho_F^{1/2} \, c \, [\![\boldsymbol{U} \cdot \boldsymbol{n}]\!] \|_{1/2,h,\Gamma},$$

and thus, using $0 \leq (\epsilon^{1/2} a - \epsilon^{-1/2} b)^2$,

$$a_h(\boldsymbol{U},\boldsymbol{U}) \geq \|\boldsymbol{U}\|_{E,h}^2 - \frac{1}{\epsilon} \|\rho_F^{1/2} \, c \nabla \cdot \boldsymbol{U}_F\|_{-1/2,h,\Gamma}^2 - \epsilon \|\rho_F^{1/2} \, c \, [\![\boldsymbol{U} \cdot \boldsymbol{n}]\!] \|_{1/2,h,\Gamma}^2.$$

Finally, invoking Lemma 1 we find

$$a_h(\boldsymbol{U},\boldsymbol{U}) \geq C \|\boldsymbol{U}\|_{E,h}^2,$$

as long as $\epsilon > C_I$ and $\gamma > \epsilon \, \rho_F \, c^2$. In the case of $\boldsymbol{v}_F \in \mathcal{R}T_0$, we thus require

$$\gamma|_E > \frac{\rho_F \, c^2 h_E^2}{|K|}$$

at each element K (on the fluid side) with edge E at Γ.

Remark 2. *There are non-zero fluid modes that do not affect the structure and which have zero energy. Thus $a_h(\cdot,\cdot)$ is only positive semi-definite (one could also say that $\|\cdot\|_{E,h}$ is a semi-norm). However, this is inherent in the physics and is not related to the stability problem obtained when choosing γ too small. Cf.* [1, 2] *for a more thorough discussion.*

3.1. **Relation to a Lagrange multiplier method.** In [1], Alonso & al. analysed a Lagrange multiplier method using piecewise constant pressures on the trace mesh $\mathcal{G}_h^1$. This method, which was shown to be stable for approximations in $\mathcal{R}T_0 \times W_1$, can be written as the problem of finding $\lambda^h \in \mathbb{R}$ and $(\boldsymbol{U}_F, \boldsymbol{U}_S, P) \in \mathcal{R}T_0 \times W_1 \times \mathcal{C}_h$, where

$$\mathcal{C}_h := \{q \in L_2(\Gamma) : q|_E \in P_0(E), \; \forall E \in \mathcal{G}_h^1\},$$

such that

$$\begin{aligned} (\rho_F c^2 \nabla \cdot \boldsymbol{U}_F, \nabla \cdot \boldsymbol{v}_F)_{\Omega_F} \;\; &+ \;\; (\boldsymbol{\sigma}(\boldsymbol{U}_S), \boldsymbol{\varepsilon}(\boldsymbol{v}_S))_{\Omega_S} + (P, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_\Gamma \\ &= \;\; (\lambda^h \rho_F \boldsymbol{U}_F, \boldsymbol{v}_F)_{\Omega_F} + (\lambda^h \rho_S \boldsymbol{U}_S, \boldsymbol{v}_S)_{\Omega_S}, \end{aligned} \tag{3.10}$$

and

$$(q, [\![\boldsymbol{U} \cdot \boldsymbol{n}]\!]) = 0, \tag{3.11}$$

for all $(\boldsymbol{v}_F, \boldsymbol{v}_S, q) \in \mathcal{R}T_0 \times W_1 \times \mathcal{C}_h$.

In order to compare our approach to this multiplier method we modify our bilinear form to

$$\begin{aligned}\tilde{a}_h(\boldsymbol{U},\boldsymbol{v}) \;:=\; & (\rho_F c^2 \nabla\cdot\boldsymbol{U}_F, \nabla\cdot\boldsymbol{v}_F)_{\Omega_F} + (\boldsymbol{\sigma}(\boldsymbol{U}_S), \boldsymbol{\varepsilon}(\boldsymbol{v}_S))_{\Omega_S} \\ & -(\rho_F c^2 \nabla\cdot\boldsymbol{U}_F, [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!])_\Gamma - (\rho_F c^2 \nabla\cdot\boldsymbol{v}_F, [\![\boldsymbol{U}\cdot\boldsymbol{n}]\!])_\Gamma \\ & +\gamma \textstyle\sum_{E\in\mathcal{G}_h^1} h_E^{-1} \left(P_h [\![\boldsymbol{U}\cdot\boldsymbol{n}]\!], P_h [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]\right)_E,\end{aligned}$$

where we have introduced $P_h [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]$ as the mean value of $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]$ on E. If we now let $\gamma\to\infty$, we see that the mean value of $[\![\boldsymbol{U}\cdot\boldsymbol{n}]\!]$ on each $E\in\mathcal{G}_h^1$ tends to zero, which is also the statement of (3.11). Since $\rho_F c^2\nabla\cdot\boldsymbol{v}_F$ is a constant, for $\boldsymbol{v}_F\in\mathcal{R}T_0$, on each edge in $\mathcal{G}_h^1$, we thus recover the solution to the Lagrange multiplier problem in the limit of infinite γ. Note that we cannot in general let $\gamma\to\infty$ in our approach; only by using a suitable projection in the penalty term is this possible. This is similar to the reduced integration penalty methods, cf. [5].

It should be noted that in our approach, the jump is not enforced to be zero. However, the vibrational modes in which the jump is large are in the upper part of the spectrum. Such modes are not to be considered spurious in the sense that they are pushed higher up in the spectrum as the mesh is refined.

4. Numerical example

To demonstrate the proposed method we solve the eigenvalue problem arising from the coupled problem given in Figure 1a. The outer domain dimensions are $1.5\times 1.5\,\mathrm{m}^2$ and the inner dimensions are $1\times 1\,\mathrm{m}^2$, and the non-matching meshes are seen in Figure 1b. The structure is discretized with both linear and quadratic triangle elements. In the simulation the following data will be used: $\rho_F = 1000\,\mathrm{kg/m}^3$, $c = 1430\,\mathrm{m/s}$, $\rho_S = 7700\,\mathrm{kg/m}^3$, $E = 144\,\mathrm{GPa}$ and $\nu = 0.35$, where E is Young's modulus and ν is Poisson's ratio. Further, the Lamé constants are given as $\mu = E/[2(1+\nu)]$ and $\lambda = E\nu/[(1+\nu)(1-2\nu)]$. The parameter $\alpha = 0$ and the penalty parameter is chosen as $\gamma = 2\rho_F c^2 h_E^2/|K|$. The computed eigenfrequencies [Hz] to the model problem are given in Table 1, and some of the corresponding eigenmodes are seen in Figure 2. The 'exact' eigenfrequencies given in Table 1 are those extrapolated eigenfrequencies given in [2]. To see how the coupling of the normal displacement on the interface works, the normal displacement of the finite element solution for three eigenmodes are plotted in Figure 3. In the figures to the left the structure is discretized using linear elements and to the right using quadratic elements. The discontinuous line is the normal displacement of the fluid and the continuous line is the structural normal displacement.

5. Concluding remarks

In this paper, we have suggested a new approach for fluid-solid coupling in fluid-structure vibration analysis. In the spirit of the finite element method, the coupling is achieved by use of a weak formulation of the interface conditions. It is possible to use for any combination of polynomial degree finite element method on either side of the interface; we have used it in combination with nonconforming finite element for the fluid and conforming finite elements for the structure, which in particular alleviates the problem of spurious zero eigenvalues.

We believe that the general methodology suggested by Nitsche's method [9], as applied in this paper, holds great promise for interface problems. To our knowledge, together with [3] this is the first example of application of the Nitsche methodology to multiphysics problems. We are currently investigating the extension to fluid-structure interaction in the space–time domain.

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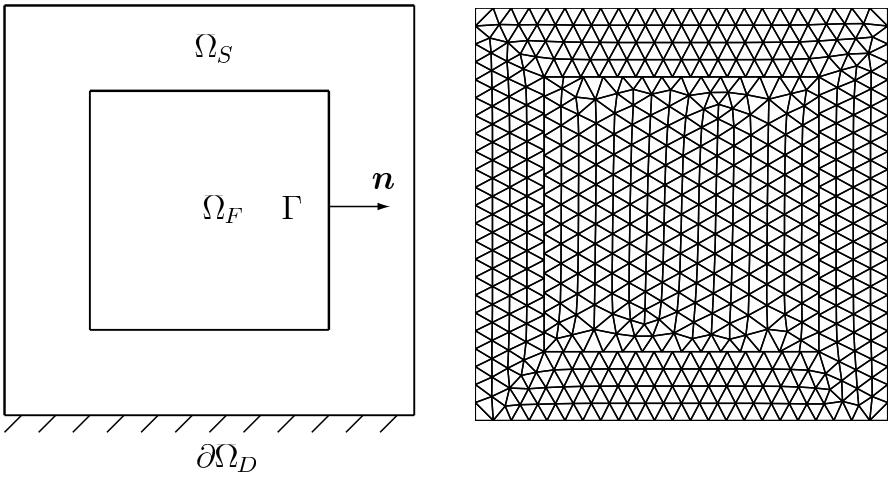

FIGURE 1. Model problem (left) and the domain with a non-matching mesh (right).

TABLE 1. The computed coupled eigenfrequencies [Hz] using linear and quadratic structural elements compared with the extrapolated eigenfrequencies in Reference [2].

Mode	Linear	Quadratic	Err. [%] (Lin.)	Err. [%] (Quad.)	'Exact'
1	113.9	106.1	11.54	3.88	102.2
2	364.5	345.0	8.22	2.44	336.8
3	577.9	548.5	13.42	7.66	509.5
4	641.3	621.3	5.93	2.62	605.4
5	681.7	672.3	1.70	0.30	670.3
6	757.6	749.0	1.54	0.39	746.1
7	833.2	822.2	1.55	0.21	820.5
8	901.1	868.0	5.12	1.26	857.2
9	1005.8	998.9	1.28	0.59	993.0

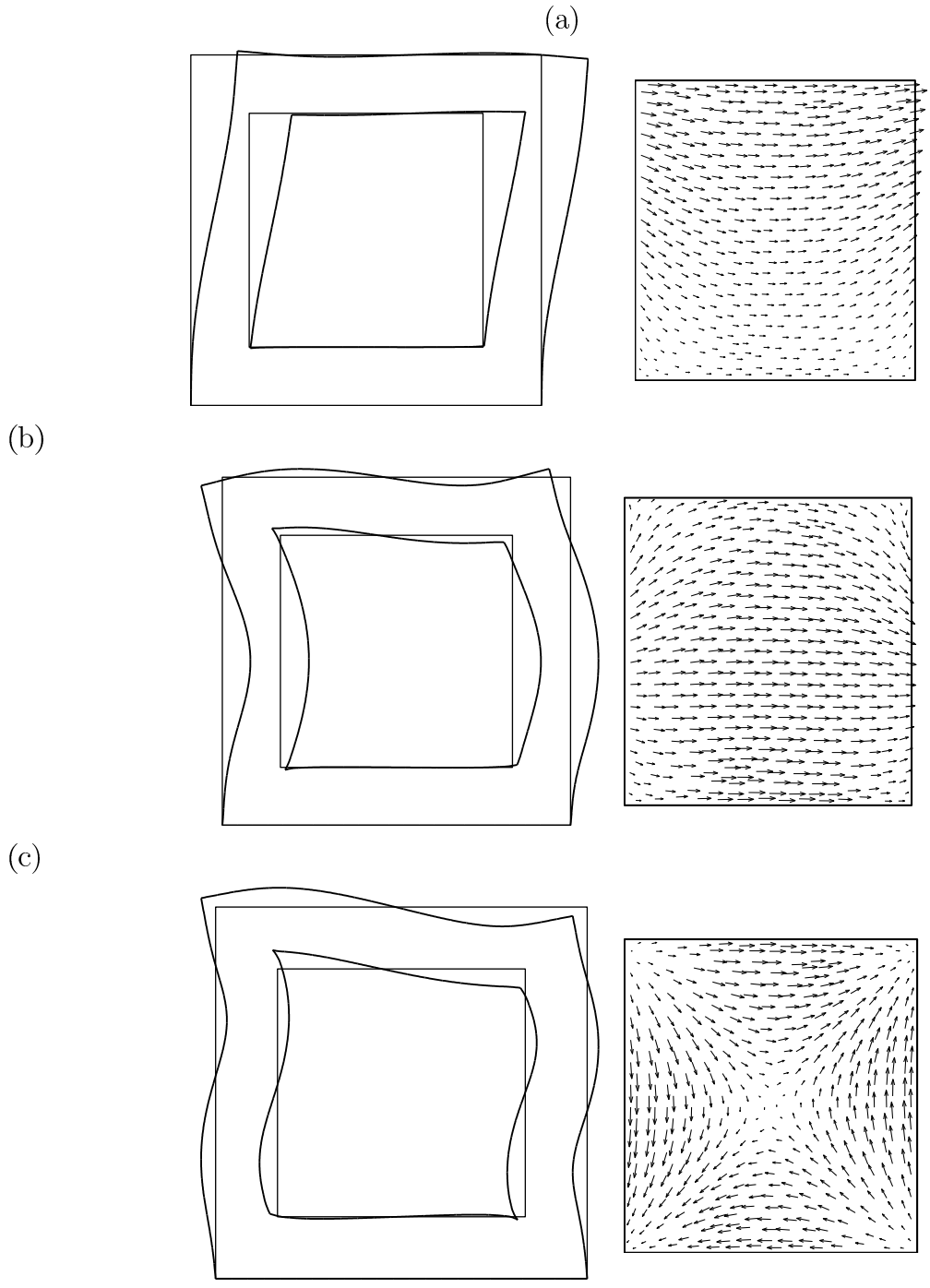

FIGURE 2. Eigenmodes of the coupled problem in Figure 1. The displaced structure to the left and the displacement field of the enclosed fluid to the right. (a) The first eigenmode, (b) the third eigenmode, and (c) the ninth eigenmode.

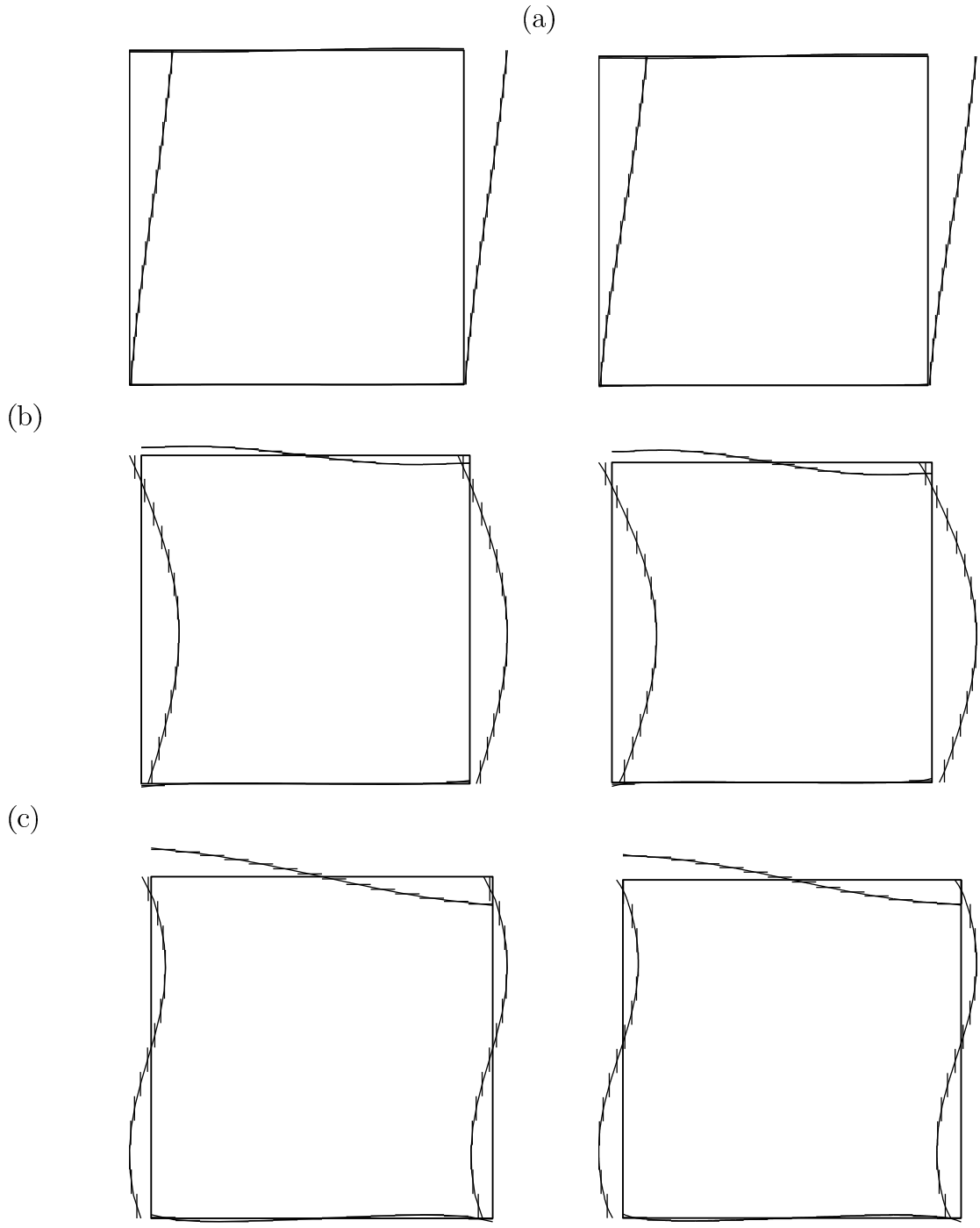

FIGURE 3. The normal displacement of the FE solution to the coupled problem on the interface. The discontinuous line is the fluid normal displacement and the continuous the structural normal displacement. Linear structural elements were used in the left figures and quadratic in the right figures. (a) The 1st, (b) the 3rd, and (c) the 9th eigenmode.

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