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POSTPROCESSING THE FINITE ELEMENT METHOD FOR SEMILINEAR PARABOLIC PROBLEMS

YUBIN YAN

ABSTRACT. In this paper we consider postprocessing of the finite element method for semilinear parabolic problems. The postprocessing amounts to solving a linear elliptic problem on a finer grid (or higher-order space) once the time integration on the coarser mesh is completed. The convergence rate is increased at almost no additional computational cost. This procedure was introduced and analyzed in Garcia-Archilla and Titi [13]. We extend the analysis to the fully discrete case and prove error estimates for both space and time discretization. The analysis is based on error estimates for the approximation of time derivatives by difference quotients.

1. INTRODUCTION

In this paper we shall consider postprocessing of the finite element method for the semilinear parabolic problem

$$u_t - \Delta u = F(u) \quad \text{in}\, \Omega, \quad \text{for}\, t \in (0,T], \qquad (1.1)$$
$$u = 0 \quad on\ \partial\Omega, \quad \text{for}\, t \in (0,T], \quad \text{with}\, u(0) = v,$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 1,2,3$, with a sufficiently smooth boundary $\partial\Omega$, $u_t = \partial u/\partial t$, Δ is the Laplacian, and $F : \mathbf{R} \to \mathbf{R}$ is a smooth function.

Let $H = L_2(\Omega)$. We define the unbounded operator $A = -\Delta$ on H with domain of definition $\mathcal{D}(A) = H^2 \cap H_0^1$, where, for integer $m \geq 1$, $H^m = H^m(\Omega)$ denotes the standard Sobolev space $W_2^m(\Omega)$, and $H_0^1 = H_0^1(\Omega) = \{v \in H^1 : v|_{\partial\Omega} = 0\}$. Then A is a closed, densely defined, and self-adjoint positive definite operator in H with compact inverse. The initial-boundary value problem (1.1) may then be formulated as the following initial value problem

$$u_t + Au = F(u), \quad \text{for}\ 0 < t \leq T, \quad \text{with}\, u(0) = v, \qquad (1.2)$$

in the Hilbert space H.

Recently, a postprocessing technique has been introduced to increase the efficiency of Galerkin method of spectral type, see Canuto, Hussaini, Quarteroni, and Zang [4], De Frutos, Garcia-Archilla, and Novo [6], De Frutos and Novo [7], [9]. Postprocessed methods yield greater accuracy than standard Galerkin schemes at nearly the same computational

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cost. In Garcia-Archilla and Titi [13], the postprocessing technique has been extended to the h-version of the finite element method for dissipative partial differential equations. There, the authors prove that the postprocessed method has a higher rate of convergence than the standard finite element method when higher order finite elements, rather than linear finite elements, are used. Error estimates in L_2 and H^1 norms in the spatially semidiscrete case are obtained. More recently, in De Frutos and Novo [8], the authors show that the postprocessing technique can also be applied to linear finite elements and the convergence rate can be improved in H^1 norm, but not in L_2 norm. The analysis is restricted to the spatially semidiscrete case.

The purpose of the present paper is to derive the error estimates in the fully discrete case for the postprocessed finite element method applied to (1.2). To do this, we introduce the time-stepping method to compute the discrete solution of (1.2) and define a difference quotient approximation to time derivative. We then define the postprocessing step in the fully discrete case and show the error estimates for postprocessing method by using the error estimates for time derivatives. For simplicity we only consider the error estimates in L_2 norm. Our technique of proof is related to, but different from, the one employed in Garcia-Archilla and Titi [13].

The paper is organized as follows. In Section 2, we introduce some basic notations and lemmas. In Section 3 we consider error estimates for the postprocessed finite element method in the semidiscrete case. In Section 4, we consider error estimates in the fully discrete case. In Section 5, we consider the starting approximation of time derivatives. Finally, in Section 6, we consider higher order time-stepping in the context of the linear homogeneous problem.

By C_0 we denote positive constant independent of the functions and parameters concerned, but not necessarily the same at different occurrences.

2. Preliminaries

Let $\mathcal{T}$ denote a partition of Ω into disjoint triangles τ such that no vertex of any triangle lies on the interior of a side of another triangle and such that the union of the triangles determine a polygonal domain $\Omega_h \subset \Omega$ with boundary vertices on $\partial\Omega$. Let h denote the maximal length of the sides of the triangulation $\mathcal{T}_h$. We assume that the triangulations are quasiuniform in the sense that the triangles of $\mathcal{T}_h$ are of essentially the same size.

Let r be any nonnegative integer. We denote by $\|\cdot\|_r$ the norm in H^r. Let $\{S_h\} = \{S_{h,r}\} \subset H_0^1$ be a family of finite element spaces with the accuracy of order $r \geq 2$, i.e., S_h consists of continuous functions on the closure $\bar{\Omega}$ of Ω which are polynomials of degree at most $r-1$ in each triangle of $\mathcal{T}_h$ and which vanish outside Ω_h, such that, for small h,

$$\inf_{\chi \in S_h} \{\|v-\chi\| + h\|\nabla(v-\chi)\|\} \leq Ch^s\|v\|_s, \quad \text{for } 1 \leq s \leq r,$$

when $v \in H^s \cap H_0^1$.

The semidiscrete problem of (1.2) is to find the approximate solution $u_h(t) = u_h(\cdot, t) \in S_h$ for each t, such that,

$$u_{h,t} + A_h u_h = P_h F(u_h), \quad \text{with } u_h(0) = v_h, \tag{2.1}$$

where $v_h \in S_h$, $P_h : L_2 \to S_h$ is the L_2 projection onto S_h, and $A_h : \ S_h \to S_h$ is the discrete analogue of A, defined by

$$(A_h\psi, \chi) = A(\psi, \chi), \quad \forall\, \psi, \chi \in S_h. \tag{2.2}$$

Here $A(\cdot,\cdot) = (\nabla\cdot, \nabla\cdot)$ is the bilinear form on H_0^1 obtained from A.

Error estimates for finite element methods for semilinear parabolic problems with various conditions on the nonlinearity have been considered in many papers, see, e.g., Akrivis, Crouzeix, and Makridakis [1], [2], Crouzeix, Thomée, and Wahlbin [5], Elliott and Larsson [10], [11], Helfrich [14], Johnson, Larsson, Thomée, and Wahlbin [15], Thomée [21], Thomée and Wahlbin [22], Wheeler [23]. The long time behavior of finite element solutions was studied by Elliott and Stuart [12], Larsson [16], [17], Larsson and Sanz-Serna [18], [19].

Let us now describe the idea of the postprocessed finite element method proposed by Garcia-Archilla and Titi [13]. Suppose that we want to obtain high order approximation, for instance $O(h^{r+2})$. Then we can use, in every time step, either a family of high order finite element spaces $\tilde{S}_h := S_{h,r+2}$ with the order $r+2$ of accuracy, or a family of finite element space $\tilde{S}_h := S_{\tilde{h},r}$ with accuracy of order r, but with finer partition $\mathcal{T}_{\tilde{h}}$ of the domain Ω, such that, $h^{r+2} = \tilde{h}^r$. In [13], another technique, called the *postprocessed finite element method*, is presented, which improves the convergence rate without using a high order finite element space $\tilde{S}_h$ in every time step. Suppose that we are interested in the solution of (1.2) at a given time T. At time T, rewriting (1.2), we have

$$Au(T) = -u_t(T) + F(u(T)). \tag{2.3}$$

Thus, $u(T)$ can be seen as the solution of an elliptic problem whose right hand side is not known but can be approximated. Garcia-Archilla and Titi first compute $u_h(T)$ by (2.1) in the finite element space S_h, then replace $u_t(T)$ by $u_{h,t}(T)$ and solve (or, in practice, approximate) the following linear elliptic problem: find $\tilde{u}(T) \in \mathcal{D}(A)$, such that,

$$A\tilde{u}(T) = -u_{h,t}(T) + F(u_h(T)), \tag{2.4}$$

which is the postprocessing step.

They obtained the following error estimate, with $\ell_h = 1 + \log(T/h^2)$,

$$\|\tilde{u}(T) - u(T)\| \le C(u)\ell_h h^{r+2}, \quad \text{for } r \ge 4, \tag{2.5}$$

where $C(u)$ is some constant depending on u. A similar result holds for $r \ge 3$. The proof is based on superconvergence for elliptic finite element methods in norms of negative order, which is the reason for the restriction $r \ge 3$.

We note that the bound (2.5) is an improvement over the error estimates for the standard Galerkin method, which is $O(h^r)$. In practice $\tilde{u}$ can not be computed exactly, since in general it does not belong to a finite element space. However, one can approximate the solution $\tilde{u}$ of (2.4) by some $\tilde{u}_h$ belonging to a finite element space $\tilde{S}_h$ of approximation order $r+2$ as described above. More precisely, we pose the following semidiscrete problem corresponding to (2.4): find $\tilde{u}_h \in \tilde{S}_h$, such that,

$$\tilde{A}_h\tilde{u}_h(T) = \tilde{P}_h\big(- u_{h,t}(T) + F(u_h(T))\big), \tag{2.6}$$

where $\tilde{P}_h : L_2 \to \tilde{S}_h$ is the L_2 projection onto $\tilde{S}_h$ and $\tilde{A}_h$ is the discrete analogue of A with respect to $\tilde{S}_h$. The standard error estimate reads, see, e.g., Brenner and Scott [3],

$$\|\tilde{u}_h(T) - \tilde{u}(T)\| \le C(u) h^{r+2}. \tag{2.7}$$

Combining (2.5) and (2.7), we have

$$\|\tilde{u}_h(T) - u(T)\| \le \|\tilde{u}_h(T) - \tilde{u}(T)\| + \|\tilde{u}(T) - u(T)\| \le C(u)\ell_h h^{r+2}, \quad \text{for } r \ge 4.$$

Let us now introduce norms of negative order. Consider the stationary problem,

$$Au = f. \tag{2.8}$$

The variational form of this problem is to find $u \in H_0^1 = H_0^1(\Omega)$, such that

$$A(u, \varphi) = (f, \varphi), \quad \forall\, \varphi \in H_0^1.$$

The standard Galerkin finite element problem is to find $u_h \in S_h$, such that,

$$A(u_h, \chi) = (f, \chi), \quad \forall\, \chi \in S_h. \tag{2.9}$$

Let $G : L_2 \to H_0^1$ be the exact solution operator of (2.8) and define the approximate solution operator $G_h : L_2 \to S_h$ by $G_h f = u_h$ so that $u_h = G_h f \in S_h$ is the solution of (2.9). We recall that G_h is the selfadjoint, positive semidefinite on L_2 and positive definite on S_h. Further we have, see Thomée [21, Chapter 6],

$$\|(G_h - G)f\| \le Ch^r \|f\|_{r-2}, \quad \text{for } f \in H^{r-2},\ r \ge 2, \tag{2.10}$$

and

$$\|(G_h - G)f\|_{-2} \le Ch^{r+2} \|f\|_{r-2}, \quad \text{for } f \in H^{r-2}, \quad r \ge 4. \tag{2.11}$$

Here r is the order of the accuracy of the family $\{S_h\}$, and the negative order norm is defined by

$$\|\cdot\|_{-2} = \sup\Big\{ \frac{(v, \phi)}{\|\phi\|_2} : \phi \in H^2 \Big\}.$$

We note that $G : L_2 \to H_0^1 \cap H^2$ is the inverse operator of $A : H_0^1 \cap H^2 \to L_2$, i.e., $G = A^{-1}$, and similarly $G_h = A_h^{-1}$ on S_h, where A_h is the discrete Laplacian of A defined by (2.2). Moreover, we will use the following properties, see, Thomée [21, Chapter 2],

$$G_h P_h = G_h \quad \text{and} \quad G_h = R_h G, \tag{2.12}$$

where $R_h : H_0^1 \to S_h$ is the elliptic projection, or Ritz projection, defined by

$$A(R_h u, \chi) = A(u, \chi), \quad \forall\, \chi \in S_h. \tag{2.13}$$

For our analysis it will be convenient to use instead of the negative order norm introduced above, such a norm defined by

$$|v|_{-s} = \|G^{s/2} v\| = (G^s v, v)^{1/2}, \quad \text{for } s \ge 0,$$

we think of this as a norm in L_2.

We introduce also a discrete negative order seminorm on L_2 by

$$|v|_{-s,h} = \|G_h^{s/2} v\| = (G_h^s v, v)^{1/2}, \quad \text{for } s \ge 0;$$

it corresponds to the discrete semi-inner product $(v,w)_{-s,h} = (G_h^s v, w)$, $\forall\, v, w \in L_2$. Since G_h is positive definite on S_h, $|v|_{-s,h}$ and $(v,w)_{-s,h}$ define a norm and an inner product there. We also find that the discrete negative order seminorm is equivalent to the corresponding continuous norm, modulo a small error. More precisely, we have the following bounds, see, e.g., Thomée [21, Lemma 6.3],

Lemma 2.1. *We have, for $0 \le s \le r$,*

$$|v|_{-s,h} \le C_0(|v|_{-s} + h^s\|v\|), \quad \textit{and} \quad |v|_{-s} \le C_0(|v|_{-s,h} + h^s\|v\|).$$

We also need Gronwall's lemma.

Lemma 2.2. *If a, b are nonnegative constants and*

$$0 \le u(t) \le a + b\int_0^t u(s)\,ds, \quad \textit{for } 0 \le t \le T,$$

then we have

$$u(t) \le ae^{bt}, \quad \textit{for } 0 \le t \le T.$$

For the nonlinear operator F, we have the following bounds, see Garcia-Archilla and Titi [13, Lemma 3]. For the sake of completeness, we include the proof, written in our slightly simpler form.

Lemma 2.3. *Let $u \in H^r(\Omega) \cap H_0^1(\Omega)$, $r \ge 4$, and $\chi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Assume that F is a smooth function. Further assume that $d \le 3$ and $\|u - \chi\|_{L_\infty} \le K$ for some positive number K. Then there is a constant $C = C(\|u\|_r, K)$ such that*

$$\|F(u) - F(\chi)\| \le C\|u - \chi\|, \tag{2.14}$$

and

$$|F(u) - F(\chi)|_{-2} \le C\big(|u - \chi|_{-2} + \|u - \chi\|^2\big). \tag{2.15}$$

Proof. By Taylor's formula, we have, with $\xi = u + \theta(\chi - u)$, $0 \le \theta \le 1$,

$$\|F(u) - F(\chi)\| = \|F'(\xi)(u - \chi)\| \le \|F'(\xi)\|_{L_\infty}\|u - \chi\|.$$

Since $F'(x)$ is bounded in $\{x : |x| \le \|u\|_{L_\infty} + \|\chi - u\|_{L_\infty}\}$, we have, noting that $\|u\|_{L_\infty} \le C_0\|u\|_r$ for $r \ge 4$, $d \le 3$,

$$\|F'(\xi)\|_{L_\infty} \le C(\|u\|_{L_\infty}, K) \le C(\|u\|_r, K), \tag{2.16}$$

which shows (2.14).

To prove (2.15), we have, by Taylor's formula,

$$|F(u) - F(\chi)|_{-2} \le |F'(u)(u - \chi)|_{-2} + \frac{1}{2}|F''(\xi)(u - \chi)^2|_{-2}.$$

We first show

$$|F''(\xi)(u - \chi)^2|_{-2} = \|A^{-1}F''(\xi)(u - \chi)^2\| \le C\|u - \chi\|^2. \tag{2.17}$$

In fact, by duality, for $\forall\, \phi \in L_2(\Omega)$,

$$\begin{aligned}\big|\big(A^{-1}F''(\xi)(u-\chi)^2, \phi\big)\big| &= \big|\big(F''(\xi)(u-\chi)^2, A^{-1}\phi\big)\big| \\ &\leq \|F''(\xi)\|_{L_\infty}\|(u-\chi)^2\|_{L_1}\|A^{-1}\phi\|_{L_\infty}.\end{aligned}$$

Following (2.16), we have $\|F''(\xi)\|_{L_\infty} \leq C(\|u\|_r, K)$. Further, by Sobolev's inequality and elliptic regularity, we have

$$\|A^{-1}\phi\|_{L_\infty} \leq C_0\|A^{-1}\phi\|_{H^2} \leq C_0\|\phi\|, \quad \text{for } d \leq 3.$$

Thus,

$$\big|\big(A^{-1}F''(\xi)(u-\chi)^2, \phi\big)\big| \leq C\|u-\chi\|^2\|\phi\|, \quad \forall\, \phi \in L_2(\Omega),$$

which implies that (2.17) holds.

Now we show

$$|F'(u)(u-\chi)|_{-2} = \|A^{-1}F'(u)(u-\chi)\| \leq C|u-\chi|_{-2}. \tag{2.18}$$

In fact, by duality, for $\forall\, \phi \in L_2(\Omega)$, noting that $F'(u)A^{-1}\phi \in \mathcal{D}(A)$,

$$\begin{aligned}\big(A^{-1}F'(u)(u-\chi), \phi\big) &= \big(F'(u)(u-\chi), A^{-1}\phi\big) \\ &= \big(A^{-1}(u-\chi), A(F'(u)A^{-1}\phi)\big).\end{aligned}$$

With $A = -\Delta$, we have

$$\begin{aligned}\|A(F'(u)A^{-1}\phi)\| &= \|F'(u)\phi + 2\nabla F'(u)\cdot\nabla(A^{-1}\phi) + (\Delta F'(u))A^{-1}\phi\| \\ &\leq \|F'(u)\|_{L_\infty}\|\phi\| + 2\|\nabla F'(u)\|_{L_\infty}\|A^{-1}\phi\|_{H^1} + \|\Delta F'(u)\|_{L_\infty}\|A^{-1}\phi\| \\ &\leq C\|F'(u)\|_{W^2_\infty}\|A^{-1}\phi\|_{H^2} \leq C(\|u\|_r)\|\phi\|.\end{aligned}$$

Thus we get

$$|(A^{-1}F'(u)(u-\chi), \phi)| \leq C(\|u\|_r)\|A^{-1}(u-\chi)\|\,\|\phi\|,$$

which implies (2.18).

Together these estimates complete the proof. □

Remark 2.1. In our application of Lemma 2.3, we will choose u to be the solution of (1.2) and χ to be the corresponding finite element approximation solution u_h. It is obvious that u_h and u satisfy the assumptions of the Lemma 2.3. For instance $\|u_h - u\|_\infty \leq K$ can be achieved by using the inverse inequality provided we know that the L_2 error estimates for $u_h - u$ is $O(h^r)$, see Thomée [21, Chapter 14].

3. Semidiscrete approximation

In this section we will consider the error estimates for the *postprocessed finite element method* for the semilinear parabolic problem (1.2) in the semidiscrete case. The main theorem in this section is the following:

Theorem 3.1. *Let $r \geq 4$ and S_h and $\tilde{S}_h$ be the finite element spaces of order r and $r+2$, respectively, as described in Section 1. Let $\tilde{u}_h$ and u be the solutions of (2.6) and (1.2), respectively. Assume that F satisfies $\|F(u)\|_r \leq C_0$ in addition to the assumptions in Lemma 2.3. Let u_h be the solution of (2.1). Assume that $v_h = R_h v$ and*

$$\sup_{s\in[0,T]} \|u_h(s) - u(s)\|_{L_\infty} \leq K,$$

and

$$\sup_{s\in[0,T]} \big(\|u(s)\|_r + \|u_t(s)\|_r + \|u_{tt}(s)\|_r\big) \leq M,$$

for some positive numbers K, M, T. Then there is a constant $C = C(K, M, T)$ such that, with $\ell_h = 1 + \log(T/h^2)$,

$$\|\tilde{u}_h(T) - u(T)\| \leq C\ell_h h^{r+2}. \tag{3.1}$$

As we mentioned in Section 2, Garcia-Archilla and Titi [13] has proved the similar results: if $\tilde{u}(T)$ and u are the solutions of (2.4) and (1.2), then

$$\|\tilde{u}(T) - u(T)\| \leq C(u)\ell_h h^{r+2}, \quad \text{for } r \geq 4, \tag{3.2}$$

where $C(u)$ is some constant depending on u.

For the comparison, let us recall the idea of their proof. By (2.4) and (1.2), it follows that

$$\begin{aligned}\|\tilde{u}(T) - u(T)\| &= \|A^{-1}(-u_{h,t}(T) + F(u_h(T))) - A^{-1}(-u_t(T) + F(u(T)))\|\\ &\leq |u_{h,t}(T) - u_t(T)|_{-2} + |F(u_h(T)) - F(u(T))|_{-2}.\end{aligned}$$

Lemma 2.3 with $u = u(T)$ and $\chi = u_h(T)$ implies that

$$|F(u_h(T)) - F(u(T)|_{-2} \leq C(\|u\|_r, K)\big(|u_h(T) - u(T)|_{-2} + \|u_h(T) - u(T)\|^2\big).$$

Introducing elliptic projection R_h defined by (2.13), it follows that

$$\begin{aligned}\|\tilde{u}(T) - u(T)\| \leq{}& |\rho_t(T)|_{-2} + |\theta_t(T)|_{-2}\\ &+ C(\|u\|_r, K)\big(|\rho(T)|_{-2} + |\theta(T)|_{-2} + \|\rho(T)\|^2 + \|\theta(T)\|^2\big),\end{aligned} \tag{3.3}$$

where $\rho = R_h u - u$, $\theta = u_h - R_h u$, $\rho_t = R_h u_t - u_t$, and $\theta_t = u_{h,t} - R_h u_t$.

The desired bounds of $|\rho|_{-l}, l = 0, 2$, and $\|\rho_t\|_{-2}$ are well known, see, e.g., Thomée [21, Chapter 6]. The task is to estimate $|\theta|_{-l}$ for $l = 0, 2$, and $|\theta_t|_{-2}$. To do this, consider the following equation

$$\theta_t + A_h\theta = P_h(-\rho_t + F(u_h) - F(u)), \quad \text{with } \theta(0) = v_h - R_h v = 0. \tag{3.4}$$

By Duhamel's principle, it follows, with $E_h(t) = e^{-tA_h}$,

$$\theta(T) = \int_0^T E_h(T-s)P_h\big(-\rho_t(T) + F(u_h(T)) - F(u(T))\big)\,ds. \tag{3.5}$$

The desired bounds of $\|\theta\|_{-l}$ for $l = 0, 2$ can be easily proved by using Lemma 2.2 and the stability of $E_h(t)$. To estimate $|\theta_t|_{-2}$, they note that, by the second part of Lemma 2.1 with $s = 2$,

$$|\theta_t|_{-2} = C_0(h^2\|\theta_t\| + |\theta_t|_{-2,h}).$$

By (3.4), we have

$$|\theta_t|_{-2,h} \le \|\theta\| + |P_h(-\rho_t + F(u_h) - F(u))|_{-2,h},$$

and, noting that $\|A_h\theta\| \le Ch^{-2}\|\theta\|$,

$$\begin{aligned}\|\theta_t\| &\le \|A_h\theta\| + \|P_h(-\rho_t + F(u_h) - F(u))\| \\ &\le Ch^{-2}\|\theta\| + \|P_h(-\rho_t + F(u_h) - F(u))\|.\end{aligned}$$

Hence

$$\begin{aligned}|\theta_t|_{-2} \le C_0\|\theta\| &+ C_0h^2\|P_h(-\rho_t + F(u_h) - F(u))\| \\ &+ C_0|P_h(-\rho_t + F(u_h) - F(u))|_{-2,h}.\end{aligned} \tag{3.6}$$

The desired bounds for the last two terms in the right hand side of (3.6) follow from Lemmas 2.1 and 2.3, and the estimates for $|\rho_t|_{-l}$ and $|u_h - u|_{-l}$, $l = 0, 2$. Further they show that θ has the superconvergence property, i.e,

$$\|\theta\| \le C(u)\ell_h h^{r+2}, \quad \text{for } r \ge 4.$$

Together these estimates completes the proof of (3.2).

We note that the logarithmic factor ℓ_h appears in the superconvergent estimate of θ.

We now return to Theorem 3.1 and state the idea of the proof in present paper. In Theorem 3.1, we consider the error bounds for $\|\tilde{u}_h - u\|$, not only for $\|\tilde{u} - u\|$. To prove Theorem 3.1, it suffices to show the bounds of $|u_h - u|_{-l}$ and $|u_{h,t} - u_t|_{-l}$ for $l = 0, 2$. We first split

$$u_h - u = (u_h - \hat{u}_h) + (\hat{u}_h - u) = \eta + e, \tag{3.7}$$

where $\hat{u}_h$ satisfies

$$\hat{u}_{h,t} + A_h\hat{u}_h = P_hF(u), \quad \hat{u}_h(0) = v_h. \tag{3.8}$$

Since u satisfies

$$u_t + Au = F(u), \quad u(0) = v, \tag{3.9}$$

the desired bounds of $e = \hat{u}_h - u$ and e_t follow from the error estimates for the linear parabolic problem because the right hand side of (3.8) is independent of $\hat{u}_h$. In other words we only need to consider the nonlinear term F when we show the bounds of $\eta = u_h - \hat{u}_h$ and η_t. Note that η satisfies

$$\eta_t + A_h\eta = P_h(F(u_h) - F(u)), \quad \eta(0) = 0. \tag{3.10}$$

By Duhamel's principle, we have

$$\eta(T) = \int_0^T E_h(T-s)P_h\big(F(u_h(s)) - F(u(s))\big)\,ds. \tag{3.11}$$

We obtain the desired bounds for $|\eta|_{-l}, l = 0, 2$, by using Lemmas 2.2 and 2.3 as above for showing the bounds of $|\theta|_{-l}, l = 0, 2$, in [13]. For $|\eta_t|_{-l}, l = 0, 2$, we have two ways to consider the bounds. One way is to use the superconvergence property of η, which can be proved as we mentioned above for proving the superconvergence property of θ in [13]. Another way is to work with the following equality

$$\begin{aligned} \eta_t(T) =& P_h(F(u_h(T)) - F(u(T))) \\ &- \int_0^T A_h E_h(T-s) P_h\big(F(u_h(s)) - F(u(s))\big)\, ds, \end{aligned} \tag{3.12}$$

which follows from (3.10) and (3.11).

Below we will use the second way to estimate $|\eta_t|_{-l}$ for $l = 0, 2$. This is the main difference between our proof and the proof in Garcia-Archilla and Titi [13]. We will extend this idea to the fully discrete case in Section 4.

We remark that since $\eta(0) = 0$, we don't need to consider the term $E_h(T)\eta(0)$ in (3.11). This observation is very useful in the fully discrete case.

Lemma 3.2. *Let u_h and u be the solutions of* (2.1) *and* (1.2), *respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_h v$ and*

$$\sup_{0\le s\le T} \|u_h(s) - u(s)\|_{L_\infty} \le K, \tag{3.13}$$

and

$$\sup_{0\le s\le T} \big(\|u(s)\|_r + \|u_t(s)\|_r\big) \le M_1, \tag{3.14}$$

for some positive numbers K, M_1, T. Then there is a constant $C = C(K, M_1, T)$ such that

$$\sup_{0\le t\le T} \|u_h(t) - u(t)\| \le Ch^r \quad \textit{for } r \ge 2, \tag{3.15}$$

and

$$\sup_{0\le t\le T} |u_h(t) - u(t)|_{-2} \le Ch^{r+2}, \quad \textit{for } r \ge 4. \tag{3.16}$$

Proof. The error estimate (3.15) is well known, see Wheeler [23] and Thomée [21], where it is proved by splitting $u_h - u = \theta + \rho$, where $\theta = u_h - R_h u, \rho = R_h u - u$. Here we will show (3.15) by splitting $u_h - u = \eta + e$, where η, e are defined by (3.7). We will use this idea in subsequent lemmas for the proof of the error estimate for time derivative approximation and later in the proof of the error estimates in fully discrete case.

For $e = \hat{u}_h - u$, we have, by the standard error estimates for linear parabolic problem in semidiscrete case, see, e.g., Thomée [21, Lemma 1.3],

$$\|e(t)\| \le \|\hat{u}_h(0) - u(0)\| + C_0 h^r \Big(\|v\|_r + \int_0^t \|u_t\|_r\, ds\Big), \quad \text{for } r \ge 2. \tag{3.17}$$

Note that $\hat{u}_h(0) = v_h = R_h v = R_h u(0)$, we therefore have

$$\|e(t)\| \le C(M_1, T) h^r, \quad \text{for } r \ge 2,\ 0 \le t \le T. \tag{3.18}$$

For $\eta = u_h - \hat{u}_h$, we have, by (3.11) and the stability of $E_h(t)$,

$$\|\eta(t)\| \leq \int_0^t \|F(u_h(s)) - F(u(s))\|\, ds.$$

By Lemma 2.3, we have

$$\|\eta(t)\| \leq C(K, M_1) \int_0^t \|u_h - u\|\, ds \leq C(K, M_1)\Big(\int_0^t \|\eta\|\, ds + \int_0^t \|e\|\, ds\Big).$$

Further, by Lemma 2.2 and (3.18),

$$\|\eta(t)\| \leq C(K, M_1) \int_0^t \|e\|\, ds \leq C(K, M_1, T) h^r, \quad \text{for } r \geq 2,\ 0 \leq t \leq T, \tag{3.19}$$

which shows (3.15).

Now we turn to (3.16). By Thomée [21, Theorem 6.2], we have, since $v_h = R_h v$,

$$|e(t)|_{-2} \leq C_0 h^{r+2} \Big(\|v\|_r + \int_0^t \|u_t\|_r\, ds\Big), \quad \text{for } r \geq 4. \tag{3.20}$$

To estimate $|\eta|_{-2}$, we first note that, by Lemma 2.1,

$$|\eta|_{-2} \leq C_0(h^2\|\eta\| + |\eta|_{-2,h}) = C_0(h^2\|\eta\| + \|G_h\eta\|). \tag{3.21}$$

Here $G_h\eta$ satisfies, by (3.10),

$$G_h\eta_t + A_h G_h \eta = G_h P_h(F(u_h) - F(u)), \quad G_h\eta(0) = 0,$$

which implies, by Duhamel's principle,

$$G_h\eta(t) = \int_0^t E_h(t-s) G_h P_h\big(F(u_h) - F(u)\big)\, ds.$$

Note that, by Lemmas 2.1 and 2.3, and (3.15), (3.20),

$$\begin{aligned}
\|G_h P_h(F(u_h) - F(u))\| &= |F(u_h) - F(u)|_{-2,h} \\
&\leq C_0\big(h^2\|F(u_h) - F(u)\| + |F(u_h) - F(u)|_{-2}\big) \\
&\leq C(\|u\|_r, K)\big(h^2\|u_h - u\| + \|u_h - u\|^2 + |u_h - u|_{-2}\big) \\
&\leq C(K, M_1, T)\big(h^{r+2} + |\eta|_{-2}\big).
\end{aligned}$$

Hence, by stability of $E_h(t)$,

$$\|G_h\eta(t)\| \leq C(K, M_1, T)\Big(h^{r+2} + \int_0^t |\eta|_{-2}\, ds\Big).$$

Combining this with (3.21), (3.19), and using Lemma 2.2, we get

$$|\eta(t)|_{-2} \leq C(K, M_1, T) h^{r+2}, \quad \text{for } 0 \leq t \leq T. \tag{3.22}$$

Together these estimates complete the proof. □

Next lemma is the error estimates for time derivative of the solution of (1.2).

Lemma 3.3. *Let u_h and u be the solutions of* (2.1) *and* (1.2)*, respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_h v$ and*

$$\sup_{0\le s\le T} \|u_h(s) - u(s)\|_{L_\infty} \le K, \tag{3.23}$$

and

$$\sup_{0\le s\le T} \big(\|u(s)\|_r + \|u_t(s)\|_r + \|u_{tt}(s)\|_r\big) \le M_2, \tag{3.24}$$

for some positive numbers K, M_2, T. Then there is a constant $C = C(K, M_2, T)$ such that, with $\ell_h = 1 + \log(T/h^2)$,

$$\sup_{0\le t\le T} \|u_{h,t}(t) - u_t(t)\| \le C\ell_h h^r, \tag{3.25}$$

and

$$\sup_{0\le t\le T} |u_{h,t}(t) - u_t(t)|_{-2} \le C\ell_h h^{r+2}. \tag{3.26}$$

Proof. We write

$$u_{h,t} - u_t = (u_{h,t} - \hat{u}_{h,t}) + (\hat{u}_{h,t} - u_t) = \eta_t + e_t.$$

Following the proofs of Theorems 1.3 and 6.2 in Thomée [21] for the error estimate $|e|_{-l},\ l = 0, 2$, we can show the following error estimates for $|e_t|_{-l},\ l = 0, 2$, that is,

$$\|e_t(t)\| \le \|\hat{u}_{h,t}(0) - u_t(0)\| + C_0 h^r \Big(\|u_t(0)\|_r + \int_0^t \|u_{tt}\|_r\, ds\Big),$$

and

$$|e_t(t)|_{-2} \le |\hat{u}_{h,t}(0) - u_t(0)|_{-2} + C_0 h^{r+2}\Big(\|u_t(0)\|_r + \int_0^t \|u_{tt}\|_r\, ds\Big).$$

We observe that, by (3.8), and noting that $\hat{u}_h(0) = R_h u(0)$,

$$\begin{aligned} \hat{u}_{h,t}(0) &= -A_h \hat{u}_h(0) + P_h F(u(0)) = -A_h R_h u(0) + P_h F(u(0)) \\ &= P_h\big(Au(0) + F(u(0))\big) = P_h u_t(0). \end{aligned}$$

We therefore have, by the error bounds for the L_2 projection,

$$\|u_{h,t}(0) - u_t(0)\| = \|(P_h - I)u_t(0)\| \le C_0 h^r \|u_t(0)\|_r,$$

and

$$|u_{h,t}(0) - u_t(0)|_{-2} \le C_0 h^{r+2} \|u_t(0)\|_r.$$

Thus, we get

$$\|e_t(t)\| \le C_0 h^r \Big(\|u_t(0)\|_r + \int_0^t \|u_{tt}\|_r\, ds\Big) \le C(M_2, T) h^r, \tag{3.27}$$

and, similarly,

$$|e_t(t)|_{-2} \le C(M_2, T) h^{r+2}. \tag{3.28}$$

We now turn to $|\eta_t|_{-l}$, $l = 0, 2$. Using the fact $\|A_h E_h(t)\| \le C_0(t + h^2)^{-1}$, see Schatz, Thomée, and Wahlbin [20], we have

$$\int_0^t \|A_h E_h(t-s)\|\, ds \le C_0(1 + \log(T/h^2)) \le C_0 \ell_h. \tag{3.29}$$

Thus, by (3.12), (3.15), and Lemma 2.3,

$$\begin{aligned}
\|\eta_t(t)\| &\le \|P_h(F(u_h(t)) - F(u(t)))\| + \int_0^t \big\|A_h E_h(t-s) P_h\big(F(u_h(s)) - F(u(s))\big)\big\|\, ds \\
&\le C(K, M_2, T)(1 + \ell_h) \sup_{0 \le s \le T} \|u_h(s) - u(s)\| \le C(K, M_2, T)\ell_h h^r,
\end{aligned}$$

For $|\eta_t(t)|_{-2}$, we have, by (3.12),

$$\begin{aligned}
|\eta_t(t)|_{-2} &\le |P_h(F(u_h(t)) - F(u(t)))|_{-2} \\
&\quad + \int_0^t \big|A_h E_h(t-s) P_h\big(F(u_h(t)) - F(u(t))\big)\big|_{-2}\, ds.
\end{aligned}$$

Here, by Lemmas 2.1 and 2.3, and (3.15), (3.16),

$$\begin{aligned}
|P_h(F(u_h) - F(u))|_{-2} &\le C_0\big(h^2\|P_h(F(u_h) - F(u))\| + \|G_h P_h(F(u_h) - F(u))\|\big) \\
&\le C(\|u\|_r, K)\big(h^2\|u_h - u\| + \|u_h - u\|^2 + |u_h - u|_{-2}\big), \\
&\le C(K, M_2, T)h^{r+2}.
\end{aligned}$$

Thus, by (3.29),

$$|\eta_t(t)|_{-2} \le C(K, M_2, T)\ell_h h^{r+2}.$$

Together these estimates complete the proof. □

Proof of Theorem 3.1. Combining (2.3) and (2.4), we have, with $\tilde{G}_h = \tilde{A}_h^{-1}$,

$$\begin{aligned}
\tilde{u}_h(T) - u(T) &= \tilde{G}_h \tilde{P}_h(-u_{h,t} + F(u_h)) - G(-u_t + F(u)) \\
&= (\tilde{G}_h \tilde{P}_h - G)\big(-u_{h,t} + F(u_h) + u_t - F(u)\big) \\
&\quad - (\tilde{G}_h \tilde{P}_h - G)(u_t - F(u)) \\
&\quad + G(-u_{h,t} + F(u_h) + u_t - F(u))
\end{aligned}$$

Thus, by Lemmas 2.3, 3.2 and 3.3, we get, noting that $\|(\tilde{G}_h \tilde{P}_h - G)f\| \le Ch^s\|f\|_{s-2}$ for $0 \le s \le r+2$,

$$\begin{aligned}
\|\tilde{u}_h(T) - u(T)\| &\le C_0 h^2\big(\|u_{h,t} - u_t\| + \|F(u_h) - F(u)\|\big) \\
&\quad + C_0 h^{r+2}\big(\|u_t\|_r + \|F(u)\|_r\big) \\
&\quad + |u_{h,t} - u_t|_{-2} + |F(u_h) - F(u)|_{-2} \\
&\le C(K, M, T)\ell_h h^{r+2}.
\end{aligned}$$

The proof is complete. □

4. Completely discrete approximation

In this section we will consider the postprocessed finite element method for (1.2) in the fully discrete case.

We use the similar technique developed in Section 3 to derive the error estimates in fully discrete case. Let $t_n = nk$, k time step. We define the following backward Euler method, with $\bar\partial U^n = (U^n - U^{n-1})/k$,

$$\bar\partial U^n + A_h U^n = P_h F(U^n), \quad n \geq 1, \quad \text{with } U^0 = v_h, \tag{4.1}$$

It is natural to approximate $u_{h,t}(T)$, $T = t_n$ in (2.4) by $\bar\partial U^n$ for fixed n. The postprocessing step in the fully discrete case is to find $\tilde u(T) \in \mathcal{D}(A)$, such that

$$A\tilde u(T) = -\bar\partial U^n + F(U^n). \tag{4.2}$$

The semidiscrete problem of (4.2) is to find $\tilde u_h(T) \in \tilde S_h$, such that,

$$\tilde A_h \tilde u_h(T) = \tilde P_h(-\bar\partial U^n + F(U^n)). \tag{4.3}$$

Let $\hat U^n$ be the solution of

$$\bar\partial \hat U^n + A_h \hat U^n = P_h F(u^n), \quad n \geq 1, \quad \text{with } \hat U^0 = v_h. \tag{4.4}$$

We have the following theorem.

Theorem 4.1. *Let $r \geq 4$ and S_h and $\tilde S_h$ be the finite element spaces of order r and $r+2$, respectively, as described in Section 1. Let $\tilde u_h$ and u be the solutions of (4.3) and (1.2), respectively. Assume that F satisfies $\|F(u^n)\|_r \leq C_0$ in addition to the assumptions in Lemma 2.3. Let $T = t_n$ be a fixed time. Let U^n be the solution of (4.1). Assume that $v_h = R_h v$ and*

$$\sup_{0\leq t_n\leq T} \|U^n - u(t_n)\|_{L_\infty} \leq K,$$

and

$$\sup_{0\leq s\leq T} \big(\|u(s)\|_r + \|u_t(s)\|_r + \|u_{tt}(s)\| + |u_{tt}(s)|_{-2} + \|Au_{tt}(s)\|\big) \leq M, \tag{4.5}$$

for some positive numbers K, M, T. Then there is a constant $C = C(K, M, T)$ such that, with $\ell_k = 1 + \log(T/k)$,

$$\|\tilde u_h(T) - u(T)\| \leq C_0\big(\|\bar\partial \hat U^1 - u_t(t_1)\| + |\bar\partial \hat U^1 - u_t(t_1)|_{-2}\big) + C\ell_k(h^{r+2} + k).$$

We now state a lemma for the error estimate of the approximation U^n of $u(t_n)$ in the L_2 norm.

Lemma 4.2. *Let U^n and u be the solutions of (4.1) and (1.2), respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_h v$, and*

$$\sup_{0\leq t_n\leq T} \|U^n - u(t_n)\|_{L_\infty} \leq K, \tag{4.6}$$

and

$$\sup_{0\leq s\leq T} \big(\|u(s)\|_r + \|u_t(s)\|_r + \|u_{tt}(s)\| + |u_{tt}(s)|_{-2}\big) \leq M_3, \tag{4.7}$$

for some positive numbers K, M_3, T. Then there is a constant $C = C(K, M_3, T)$ such that

$$\sup_{0 \le t_n \le T} \|U^n - u(t_n)\| \le C(h^r + k), \tag{4.8}$$

and

$$\sup_{0 \le t_n \le T} |U^n - u(t_n)|_{-2} \le C(h^{r+2} + k). \tag{4.9}$$

Proof. We split

$$U^n - u(t_n) = (U^n - \hat{U}^n) - (\hat{U}^n - u(t_n)) = \eta^n + e^n,$$

where $\hat{U}^n$ is defined by (4.4).

For $e^n = \hat{U}^n - u(t_n)$, we have, by the standard error estimates for linear parabolic problems, see, e.g., Thomée [21, Theorem 1.5],

$$\begin{aligned} \|e^n\| &\le C_0 \|R_h v - v\| + C_0 h^r \Big(\|v\|_r + \int_0^{t_n} \|u_t\|_r \, ds \Big) + C_0 k \int_0^{t_n} \|u_{tt}(s)\| \, ds \\ &\le C(M_3, T)(h^r + k). \end{aligned} \tag{4.10}$$

For $\eta^n = U^n - \hat{U}^n$, noting that, by (4.4) and (4.1),

$$\left\{ \begin{array}{ll} \bar{\partial}\eta^n + A_h \eta^n = P_h(F(U^n) - F(u^n)), & \text{for } n \ge 1, \\ \eta^0 = 0, & \end{array} \right. \tag{4.11}$$

we have, by Lemma 2.3, with $r(\lambda) = 1/(1+\lambda)$,

$$\begin{aligned} \|\eta^n\| &\le k \sum_{j=1}^n \|r(kA_h)^{n-j+1}\| \, \|P_h(F(U^j) - F(u^j))\| \\ &\le C_0 k \sum_{j=1}^n \|F(U^j) - F(u^j)\| \le C(K, M_3) \Big(k \sum_{j=1}^n \|\eta^j\| + k \sum_{j=1}^n \|e^j\| \Big). \end{aligned}$$

Further, by the discrete Gronwall's lemma, and (4.10), we have

$$\|\eta^n\| \le C(K, M_3, T)(h^r + k),$$

which shows (4.8).

Now we turn to (4.9). Following the proof of (4.10), we can show that,

$$\begin{aligned} |e^n|_{-2} &\le C_0 |R_h v - v|_{-2} + C_0 h^{r+2} \Big(\|v\|_r + \int_0^t \|u_t\|_r \, ds \Big) \\ &\quad + C_0 k \int_0^{t_n} |u_{tt}(s)|_{-2} \, ds \\ &\le C(M_3, T)(h^{r+2} + k). \end{aligned} \tag{4.12}$$

To estimate $|\eta^n|_{-2}$, we first note that, by Lemma 2.1,

$$|\eta^n|_{-2} \le C_0 (h^2 \|\eta^n\| + \|G_h \eta^n\|). \tag{4.13}$$

Here $G_h\eta^n$ satisfies, by (4.11),

$$\begin{cases} \bar\partial(G_h\eta^n) + A_h(G_h\eta^n) = G_hP_h(F(U^n) - F(u^n)), \quad \text{for } n \ge 1, \\ \eta^0 = 0, \end{cases} \tag{4.14}$$

which implies

$$G_h\eta^n = k\sum_{j=1}^n r(kA_h)^{n-j+1} G_hP_h(F(U^j) - F(u^j)).$$

Note that, by Lemmas 2.1 and 2.3,

$$\begin{aligned} \|G_hP_h(F(U^j) - F(u^j))\| &= |F(U^j) - F(u^j)|_{-2,h} \\ &\le C(\|u\|_r, K)\big(h^2\|U^j - u^j\| + \|U^j - u^j\|^2 + |U^j - u^j|_{-2}\big). \end{aligned}$$

Hence, by the stability of $r(\lambda)$,

$$\|G_h\eta^n\| \le C(K, M_3)\Big(k\sum_{j=1}^n |\eta^j|_{-2} + h^2k\sum_{j=1}^n \|U^j - u^j\| + k\sum_{j=1}^n (\|U^j - u^j\|^2 + |e^j|_{-2})\Big).$$

Combining this with (4.13) and using the discrete Gronwall's lemma, we get, by (4.8) and (4.12),

$$|\eta^n|_{-2} \le C(K, M_3, T)(h^{r+2} + k). \tag{4.15}$$

Together these estimates complete the proof. □

We also need the following lemma for the error estimate of the approximation $\bar\partial U^n$ of $u_t(t_n)$.

Lemma 4.3. *Let U^n and u be the solutions of (4.1) and (1.2), respectively. Assume that F satisfies the assumptions in Lemma 2.3. Further assume that $v_h = R_hv$ and*

$$\sup_{0\le t_n\le T} \|U^n - u(t_n)\|_{L_\infty} \le K, \tag{4.16}$$

and

$$\sup_{0\le s\le T} \big(\|u(s)\|_r + \|u_t(s)\|_r + \|u_{tt}(s)\|_r + \|u_{tt}(s)\| + \|Au_{tt}(s)\|\big) \le M_4, \tag{4.17}$$

for some positive numbers K, M_4, T. Then there is a constant $C = C(K, M_4, T)$ such that, with $\ell_k = 1 + \log(T/k)$,

$$\sup_{k\le t_n\le T} \|\bar\partial U^n - u_t(t_n)\| \le C_0\|\bar\partial \hat U^1 - u_t(t_1)\| + C\ell_k(h^r + k), \tag{4.18}$$

and

$$\sup_{k\le t_n\le T} |\bar\partial U^n - u_t(t_n)|_{-2} \le C_0|\bar\partial \hat U^1 - u_t(t_1)|_{-2} + C\ell_k(h^{r+2} + k). \tag{4.19}$$

Proof. We use the same notation as in Lemma 4.2 and write

$$\begin{aligned}\bar\partial U^n - u_t(t_n) &= (\bar\partial U^n - \bar\partial \hat U^n) + (\bar\partial \hat U^n - u_t(t_n)) \\ &= \bar\partial \eta^n + \big(\bar\partial \hat U^n - u_t(t_n)\big).\end{aligned}$$

We first show

$$\begin{aligned}\|\bar\partial \hat U^n - u_t(t_n)\| &\le C_0 \|\bar\partial \hat U^1 - u_t(t_1)\| + C_0 h^r \Big(\|u_t(0)\|_r + \int_0^{t_n} \|u_{tt}\|_r \, ds \Big) \\ &\quad + C_0 k \int_0^{t_n} \|A u_{tt}(s)\| \, ds \\ &\le C_0 \|\bar\partial \hat U^1 - u_t(t_1)\| + C(M_4, T)(h^r + k).\end{aligned} \tag{4.20}$$

To show (4.20), we write

$$\bar\partial \hat U^n - u_t(t_n) = (\bar\partial \hat U^n - R_h u_t(t_n)) + (R_h u_t(t_n) - u_t(t_n)) = \theta^n + \rho^n.$$

In the standard way ρ^n is bounded as desired, and it remains to consider $\theta^n \in S_h$. We have

$$\bar\partial \theta^n + A_h \theta^n = P_h \omega^n, \quad \text{for } n \ge 2,$$

where

$$\omega^n = (R_h - I)\bar\partial u_t(t_n) + A(\bar\partial u^n - u_t^n) = \sigma^n + \tau^n.$$

By stability estimate, see, e.g., Thomée [21, Theorem 10.2],

$$\|\theta^n\| \le C_0 \|\theta^1\| + C_0 k \sum_{j=2}^n \|\sigma^j\| + C_0 k \sum_{j=2}^n \|\tau^j\|, \quad \text{for } n \ge 2. \tag{4.21}$$

We have

$$k\|\sigma^n\| \le C_0 h^r \int_{t_{n-1}}^{t_n} \|u_{tt}\|_r \, ds,$$

and

$$k\|\tau^n\| \le C_0 k \|A(\bar\partial u^n - u_t^n)\| \le C_0 k \int_{t_{n-1}}^{t_n} \|A u_{tt}(s)\| \, ds.$$

Together with $\|\theta^1\| \le \|\bar\partial U^1 - u_t^1\| + \|\rho^1\|$, with the obvious bounds for $\|\rho^1\|$, this completes the proof of (4.20).

For $\|\bar\partial \eta^n\|$, we have, by (4.11),

$$\bar\partial \eta^n = P_h(F(U^n) - F(u^n)) - k \sum_{j=1}^n A_h r(kA_h)^{n-j+1} P_h(F(U^n) - F(u^n)). \tag{4.23}$$

Using the following smoothing property

$$k \sum_{j=1}^n \|A_h r(kA_h)^{n-j+1}\| \le C_0 \ell_k, \tag{4.24}$$

which follows from

$$k\sum_{j=1}^{n}\|A_h r(kA_h)^{n-j+1}\| \le C_0 k\sum_{j=1}^{n} t_{n-j+1}^{-1} = C_0\Big(1+\sum_{j=1}^{n-1} t_{n-j+1}^{-1}\Big)$$
$$\le C_0\Big(1+\int_{t_1}^{t_n}\frac{1}{s}\,ds\Big) \le C_0(1+\log(t_n/k)) \le C_0\ell_k,$$

we have, by Lemma 2.3, and (4.8),

$$\|\bar\partial\eta^n\| \le C(K,M_4)\big(\|U^n-u^n\| + \ell_k \max_{1\le j\le n}\|U^j-u^j\|\big)$$
$$\le C(K,M_4,T)\ell_k(h^r+k). \tag{4.25}$$

Together these estimates complete the proof of (4.18).

Now we turn to estimate (4.19). Following the proof of (4.20), we can show

$$|\bar\partial\hat U^n - u_t(t_n)|_{-2} \le C_0|\bar\partial\hat U^1 - u_t(t_1)|_{-2} + C_0h^{r+2}\Big(\|u_t(0)\|_r + \int_0^{t_n}\|u_{tt}\|_r\,ds\Big) \tag{4.26}$$
$$+ C_0k\int_0^{t_n}\|u_{tt}(s)\|\,ds,$$
$$\le C_0|\bar\partial\hat U^1 - u_t(t_1)|_{-2} + C(M_4,T)(h^{r+2}+k).$$

For $|\bar\partial\eta^n|_{-2}$, we have, using (4.23), and by Lemmas 2.1 and 2.3,

$$|\bar\partial\eta^n|_{-2} \le C(K,M_4,T)\ell_k \max_{1\le j\le n}\big(h^2\|U^j-u^j\| + \|U^j-u^j\|^2 + |U^j-u^j|_{-2}\big).$$

Thus, by (4.8) and (4.9),

$$|\bar\partial\eta^n|_{-2} \le C(K,M_4,T)\ell_k(h^{r+2}+k). \tag{4.27}$$

Together these estimates complete the proof. ☐

Proof of Theorem 4.1. Combining (2.3) and (4.3), we have, with $\tilde G_h = \tilde A_h^{-1}$,

$$\tilde u_h(T) - u(T) = \tilde G_h\tilde P_h(-\bar\partial U^n + F(U^n)) - G(-u_t(t_n)+F(u^n))$$
$$= (\tilde G_h\tilde P_h - G)\big(-\bar\partial U^n + F(U^n) + u_t(t_n) - F(u^n)\big)$$
$$- (\tilde G_h\tilde P_h - G)\big(u_t(t_n) - F(u^n)\big)$$
$$+ G\big(-\bar\partial U^n + F(U^n) + u_t(t_n) - F(u^n)\big).$$

Thus, we get, noting that $\|(\tilde G_h\tilde P_h - G)f\| \le Ch^s\|f\|_{s-2}$ for $0\le s\le r+2$,

$$\|\tilde u_h(T) - u(T)\| \le C_0h^2\big(\|\bar\partial U^n - u_t(t_n)\| + \|F(U^n)-F(u^n)\|\big)$$
$$+ C_0h^{r+2}\|u_t(t_n) - F(u^n)\|_r$$
$$+ |\bar\partial U^n - u_t(t_n)|_{-2} + |F(U^n)-F(u^n)|_{-2}.$$

Combining this with Lemmas 2.3, 4.2, and 4.3, we complete the proof. ☐

5. Error estimate for starting approximation of time derivative

In this section we will consider the error estimate for starting approximation of time derivative $|\bar{\partial}\hat{U}^1 - u_t(t_1)|_{-s},\ s = 0, 2$, which appears in Theorem 4.1, where u and $\hat{U}^1$ satisfy

$$u_t + Au = F(u), \quad \text{with } u(0) = v, \tag{5.1}$$

and

$$\bar{\partial}\hat{U}^1 + A_h\hat{U}^1 = P_hF(u^1), \quad \text{with } \hat{U}^0 = v_h = R_hv, \tag{5.2}$$

respectively.

The semidiscrete problem of (5.1) is to find $\hat{u}_h \in S_h$ such that,

$$\hat{u}_{h,t} + A_h\hat{u}_h = P_hF(u), \quad \text{with } \hat{u}_h(0) = R_hv. \tag{5.3}$$

We observe that we use $F(u^1)$ in (5.2), thus $|\bar{\partial}\hat{U}^1 - u_t(t_1)|_{-s},\ s = 0, 2$, can be bounded by the standard technique for nonhomogeneous linear parabolic problems. We have the following theorem:

Theorem 5.1. *Let $\hat{U}^1$ and u be the solutions of* (5.2) *and* (5.1)*, respectively. Assume that F is continuously differentiable and*

$$\|Au_t(0)\| + \|u_t(0)\|_r + \max_{0\le\tau\le k}\big(\|F'(u(\tau))u_t(\tau)\| + \|u_{tt}(\tau)\|_r\big) \le M_0,$$

for some positive number M_0. Then there is a constant $C = C(M_0)$ such that

$$\|\bar{\partial}\hat{U}^1 - u_t(t_1)\| \le C(h^r + k), \tag{5.4}$$

and

$$|\bar{\partial}\hat{U}^1 - u_t(t_1)|_{-2} \le C(h^{r+2} + k). \tag{5.5}$$

Proof. We first show (5.4). We write

$$\bar{\partial}\hat{U}^1 - u_t(t_1) = \big(\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)\big) + \big(\hat{u}_{h,t}(t_1) - u_t(t_1)\big).$$

By (3.27), we have

$$\|\hat{u}_{h,t}(t_1) - u_t(t_1)\| \le C_0h^r\Big(\|u_t(0)\|_r + \int_0^{t_1}\|u_{tt}(s)\|_r\,ds\Big). \tag{5.6}$$

For $\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1)$, we have, by (5.2) and (5.3),

$$\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1) = A_h(\hat{U}^1 - \hat{u}_h^1).$$

Here, by Taylor's formula, with $r(\lambda) = 1/(1+\lambda)$, $E_h(t) = e^{-tA_h}$,

$$\begin{aligned}
\hat{U}^1 - \hat{u}_h^1 =& \big(r(kA_h) - E_h(t_1)\big) R_h v + kr(kA_h) P_h F(u^1) \\
& - \int_0^{t_1} E_h(t_1 - s) P_h F(u(s))\, ds \\
=& \big(r(kA_h) - E_h(t_1)\big) R_h v \\
& + kr(kA_h)\big(P_h F(u(0)) + \int_0^k P_h F'(u(\tau)) u_t(\tau)\, d\tau\big) \\
& - k \int_0^1 e^{-(1-s)kA_h} \big(P_h F(u(0)) + \int_0^{ks} P_h F'(u(\tau)) u_t(\tau)\, d\tau\big)\, ds \\
=& \big(r(kA_h) - E_h(t_1)\big) R_h v + kb_0(kA_h) P_h F(u(0)) + kR(F),
\end{aligned}$$

where

$$b_0(\lambda) = r(\lambda) - \int_0^1 e^{-(1-s)\lambda}\, ds,$$

and

$$\begin{aligned}
R(F) = & \, r(kA_h) \int_0^k P_h F'(u(\tau)) u_t(\tau)\, d\tau \\
& - \int_0^1 e^{-(1-s)kA_h} \int_0^{ks} P_h F'(u(\tau)) u_t(\tau)\, d\tau\, ds.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1) =& \big(r(kA_h) - E_h(t_1)\big) A_h R_h v \\
& + kA_h b_0(kA_h) P_h F(u(0)) + kA_h R(F).
\end{aligned}$$

Noting that $A_h R_h = P_h A$ and $\lambda b_0(\lambda) = -\big(r(\lambda) - e^{-\lambda}\big)$, we get

$$\begin{aligned}
\bar{\partial}\hat{U}^1 - \hat{u}_{h,t}(t_1) &= \big(r(kA_h) - E_h(t_1)\big) P_h \big(Av - F(u(0))\big) + kA_h R(F) \\
&= \big(r(kA_h) - E_h(t_1)\big) P_h u_t(0) + kA_h R(F) \\
&= I + II.
\end{aligned} \tag{5.7}$$

For I, we have, by the error estimate for homogeneous parabolic problems,

$$\begin{aligned}
\|I\| &\le \big\| \big(r(kA_h) - E_h(t_1)\big)(P_h - R_h) u_t(0) \big\| + \big\| \big(r(kA_h) - E_h(t_1)\big) R_h u_t(0) \big\| \\
&\le \|(P_h - R_h) u_t(0)\| + C_0 k \|A_h R_h u_t(0)\| \\
&\le C_0 h^r \|u_t(0)\|_r + C_0 k \|A u_t(0)\|.
\end{aligned}$$

For II, we write

$$\begin{aligned} II &= kA_h r(kA_h) \int_0^k P_h F'(u(\tau)) u_t(\tau)\, d\tau \\ &\quad - kA_h \int_0^1 e^{-(1-s)kA_h} \int_0^{ks} P_h F'(u(\tau)) u_t(\tau)\, d\tau\, ds \\ &= II_1 + II_2. \end{aligned}$$

We have, noting that $|\lambda r(\lambda)| \le 1$, $\|P_h\| \le 1$,

$$\begin{aligned} \|II_1\| &\le \|kA_h r(kA_h)\| \int_0^k \|P_h F'(u(\tau)) u_t(\tau)\|\, d\tau \\ &\le k \max_{0\le\tau\le k} \|F'(u(\tau)) u_t(\tau)\|, \end{aligned}$$

and, by exchanging the integral order and noting that $\int_\epsilon^1 \lambda e^{-(1-s)\lambda}\, ds \le 1$, for $0 \le \epsilon \le 1$,

$$\begin{aligned} \|II_2\| &= \Big\| kA_h \int_0^k P_h F'(u(\tau)) u_t(\tau) \int_{\tau/k}^1 e^{-(1-s)kA_h}\, ds d\tau \Big\| \\ &\le k \max_{0\le\tau\le k} \|F'(u(\tau)) u_t(\tau)\| \Big\| kA_h \int_{\tau/k}^1 e^{-(1-s)kA_h}\, ds \Big\| \\ &\le k \max_{0\le\tau\le k} \|F'(u(\tau)) u_t(\tau)\|. \end{aligned}$$

Together these estimates show

$$\|\bar\partial \hat U^1 - \hat u_{h,t}(t)\| \le C_0\big(h^r \|u_t(0)\|_r + k\|Au_t(0)\| + k \max_{0\le\tau\le k} \|F'(u(\tau))u_t(\tau)\|\big). \tag{5.8}$$

Combining this with (5.6) shows (5.4).

We now turn to (5.5). We again write

$$\bar\partial \hat U^1 - u_t(t_1) = \big(\bar\partial \hat U^1 - \hat u_{h,t}(t_1)\big) + \big(\hat u_{h,t}(t_1) - u_t(t_1)\big).$$

The desired bound for $|\hat u_{h,t}(t_1) - u_t(t_1)|_{-2}$ follows from (3.28).

For $\bar\partial \hat U^1 - \hat u_{h,t}(t_1)$, we have, by Lemma 2.1,

$$|\bar\partial \hat U^1 - \hat u_{h,t}(t_1)|_{-2} \le C_0\big(h^2 \|\bar\partial \hat U^1 - \hat u_{h,t}(t_1)\| + |\bar\partial \hat U^1 - \hat u_{h,t}(t_1)|_{-2,h}\big).$$

Thus, by (5.7),

$$|\bar\partial \hat U^1 - \hat u_{h,t}(t_1)|_{-2,h} \le |I|_{-2,h} + |II|_{-2,h}.$$

For $|I|_{-2,h}$, we have, by the error estimate for homogeneous parabolic problems, see [24],

$$|I|_{-2,h} = |(r(kA_h) - E_h(t_1))P_h u_t(0)|_{-2,h} \le C_0\big(h^{r+2}\|u_t(0)\|_r + k\|Au_t(0)\|\big).$$

For $|II|_{-2,h}$, we have, noting that $|r(\lambda)| \le 1$, $\int_\epsilon^1 e^{-(1-s)\lambda}\, ds \le 1$ for $0 \le \epsilon \le 1$,

$$\begin{aligned} |II|_{-2,h} \le & \int_0^k \|kr(kA_h)\| \, \|P_h F'(u(\tau))u_t(\tau)\| \, d\tau \\ & + \Big\| k \int_0^k P_h F'(u(\tau))u_t(\tau) \int_{\tau/k}^1 e^{-(1-s)kA_h} \, ds \, d\tau \Big\| \\ \le & k^2 \max_{0\le\tau\le k} \|F'(u(\tau))u_t(\tau)\|. \end{aligned}$$

Hence we get

$$|\bar\partial \hat U^1 - \hat u_{h,t}(t_1)|_{-2,h} \le C_0 \big(h^{r+2} \|u_t(0)\|_r + k\|Au_t(0)\| + k^2 \max_{0\le\tau\le k} \|F'(u(\tau))u_t(\tau)\| \big).$$

Combining this with (5.8) shows (5.5).

Together these estimates complete the proof of the theorem. □

6. High order time-stepping

The postprocessing requires very accurate time-stepping in order to match the high order spatial approximation. It would be natural then to use a time-stepping method of higher order than the backward Euler method of Section 4. However, we have not been able to analyze such methods except in the case of linear homogeneous problems, where we can apply the analysis of time derivative approximation from [24].

In this section we consider the linear homogeneous parabolic problem

$$u_t + Au = 0, \quad \text{for } t > 0, \quad \text{with } u(0) = v. \tag{6.1}$$

We define the following time-stepping method

$$U^n = r(kA_h)U^{n-1}, \quad U^0 = v_h, \tag{6.2}$$

where $r(\lambda)$ is a rational function and accurate of order $p \ge 1$, i.e.,

$$r(\lambda) - e^{-\lambda} = O(\lambda^{p+1}), \qquad \lambda \to 0.$$

For example, if $r(\lambda) = 1/(1+\lambda)$, we have $(1 + kA_h)U^n = U^{n-1}$, which is the backward Euler method. If $r(\lambda) = (1 - \lambda/2)/(1 + \lambda/2)$, we have $(1 + \frac{1}{2}kA_h)U^n = (1 - \frac{1}{2}kA_h)U^{n-1}$ which is the Crank-Nicolson method.

Further we define the quotient $Q_k U^n$ to approximate the time derivative $u_{h,t}(t_n)$, with positive integers m_1, m_2, and real numbers c_ν,

$$Q_k U^n = k^{-1} \sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu}, \quad \text{for } n \ge m_1, \tag{6.3}$$

We assume that the operator Q_k satisfies, for any smooth function u,

$$Q_k u^n - u_t(t_n) = O(k^p), \qquad k \to 0. \tag{6.4}$$

For example,

$$Q_k u^n = \bar\partial u^n = (u^n - u^{n-1})/k, \quad \text{for } n \ge 1,$$

and

$$Q_k u^n = (\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-1})/k, \quad \text{for } n \geq 2,$$

satisfy

$$\bar{\partial} u^n - u_t(t_n) = O(k), \quad k \to 0,$$

and

$$(\tfrac{3}{2}u^n - 2u^{n-1} + \tfrac{1}{2}u^{n-1})/k - u_t(t_n) = O(k^2), \quad k \to 0,$$

respectively.

The postprocessing step in fully discrete case is to find $\tilde{u}(T) \in S_h$, $T = t_n$, such that

$$A\tilde{u}(T) = -Q_k U^n. \tag{6.5}$$

The finite element solution of the elliptic problem (6.5) with respect to $\tilde{S}_h$ is to find $\tilde{u}_h(T) \in \tilde{S}_h$, such that,

$$\tilde{A}_h \tilde{u}_h(T) = \tilde{P}_h(-Q_k U^n). \tag{6.6}$$

Our main theorem in this section is the following:

Theorem 6.1. *Let $r \geq 4$ and S_h and $\tilde{S}_h$ be the finite element spaces of order r and $r+2$, respectively, as described in Section 1. Let $\tilde{u}_h$ and u be the solutions of (6.6) and (6.1), respectively. Let $T = t_n$ be a fixed time. Then we have, if $v_h = R_h v$,*

$$\|\tilde{u}_h(T) - u(T)\| \leq C_0\big(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)} + h^{r+2}\|u_t(T)\|_r\big), \quad \text{for } r \geq 4.$$

Recalling the proof of Theorem 4.1, we note that Theorem 6.1 follows once we have proved appropriate estimates of $\|Q_k U^n - u_t(t_n)\|$ and $|Q_k U^n - u_t(t_n)|_{-2}$ which are given in the following two Lemmas.

Lemma 6.2. *Let U^n and u be the solutions of (6.2) and (6.1), respectively. Assume that $|r(\lambda)| < 1$ for $\lambda > 0$. Then, we have, if $v_h = R_h v$,*

$$\|Q_k U^n - u_t(t_n)\| \leq C_0(h^r|v|_{r+2} + k^p|v|_{2(p+1)}).$$

Lemma 6.2 was proved in [24].

Lemma 6.3. *Let U^n and u be the solutions of (6.2) and (6.1), respectively. Assume that $|r(\lambda)| < 1$ for $\lambda > 0$. Then, we have, if $v_h = R_h v$,*

$$|Q_k U^n - u_t(t_n)|_{-2} \leq C_0\big(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}\big).$$

Let us first prove the following error estimate for the approximation U^n of $u(t_n)$ in negative order norm. We do not need it here but it serves as a guide for the proof of Lemma 6.3. We remark that we choose $v_h = P_h v$, not $R_h v$.

Lemma 6.4. *Let U^n and u be the solutions of (6.2) and (6.1), respectively. Assume that $|r(\lambda)| < 1$ for $\lambda > 0$. Then, we have, if $v_h = P_h v$,*

$$|U^n - u(t_n)|_{-2} \leq C\big(h^{r+2}|v|_r + k^p|v|_{2p}\big).$$

Proof. By Thomée [21, Theorem 6.2], we have

$$|u_h(t) - u(t)|_{-2} \le C_0 h^{r+2} |v|_r.$$

Therefore it suffices to show

$$|U^n - u_h(t_n)|_{-2} = C_0\big(h^{r+2}|v|_r + k^p|v|_{2p}\big), \tag{6.7}$$

which we will prove now.

By Lemma 2.1, we have

$$|U^n - u_h(t_n)|_{-2} \le C_0(h^2\|U^n - u_h(t_n)\| + |U^n - u_h(t_n)|_{-2,h}).$$

We first estimate $|U^n - u_h(t_n)|_{-2,h} = \|G_h(U^n - u_h(t_n))\|$. We write

$$U^n - u_h(t_n) = \big(r(kA_h)^n - e^{-knA_h}\big)P_h v = F_n(kA_h)P_h v,$$

where $F_n(\lambda) = r(\lambda)^n - e^{-n\lambda}$. We need to show

$$\|G_h F_n(kA_h)P_h v\| \le C\big(h^{r+2}|v|_r + k^p|v|_{2p}\big).$$

To do this we set

$$v_k = \sum_{k\lambda_l \le 1} (v, \varphi_l)\varphi_l,$$

where φ_l and λ_l are the eigenfunctions and eigenvalues of the operator A. Then $v_k \in \dot{H}^s$ for each $s \ge 0$. Further, by the definition of the norm in $\dot{H}^s$, we find easily

$$\|v - v_k\| \le k^p |v|_{2p}, \tag{6.8}$$

$$|v_k|_{2p} \le |v|_{2p}, \tag{6.9}$$

and

$$|v_k|_{r+2j} \le k^{-j}|v|_r, \quad \text{for } 0 \le j \le p-1. \tag{6.10}$$

Applying now the identity

$$v = \sum_{j=0}^{p-1} G_h^j (G - G_h) A^{j+1} v + G_h^p A^p v, \quad \text{for } v \in \dot{H}^{2p}, \quad \text{where } G_h^0 = I, \tag{6.11}$$

to v_k, we have, with $F_n = F_n(kA_h)P_h$,

$$G_h F_n v_k = \sum_{j=0}^{p-1} G_h F_n G_h^j (G - G_h) A^{j+1} v_k + G_h F_n G_h^p A^p v_k. \tag{6.12}$$

It is easy to show that, see, e.g., Thomée [21, Lemma 7.2],

$$\|F_n(kA_h)P_h G_h^j\| \le C_0 k^j \quad \text{for } 0 \le j \le p,\ n \ge 0. \tag{6.13}$$

Thus, by (6.9) and noting the boundedness of G_h,

$$\begin{aligned}\|G_h F_n G_h^p A^p v_k\| &\le \|F_n G_h^p A^p v_k\| \le C_0 k^p \|A^p v_k\| \\ &\le C_0 k^p |v_k|_{2p} \le C_0 k^p |v|_{2p}.\end{aligned}$$

Further, by (6.10), (6.13), and using (2.11), and noting that $P_h G_h^j = G_h^j$, with $0 \le j \le p-1$,

$$\begin{aligned}\|G_h F_n G_h^j (G-G_h) A^{j+1} v_k\| &\le C_0 k^j \|G_h (G-G_h) A^{j+1} v_k\| \\ &\le C_0 k^j h^2 \big\|(G-G_h)(A^{j+1} v_k)\big\| + C_0 k^j \big|(G-G_h)(A^{j+1} v_k)\big|_{-2} \\ &\le C_0 k^j h^{r+2} \|A^{j+1} v_k\|_{r-2} \le C_0 k^j h^{r+2} |v_k|_{r+2j} \le C_0 h^{r+2} |v|_r.\end{aligned}$$

Together these estimates imply

$$\|G_h F_n v_k\| \le C_0 \big(h^{r+2}|v|_r + k^p |v|_{2p}\big).$$

Since obviously, by (6.8), the boundedness of G_h and stability, we get

$$\|G_h F_n (v - v_k)\| \le \|F_n (v-v_k)\| \le 2\|(v-v_k)\| \le C_0 k^p |v|_{2p},$$

so that

$$\big\|G_h(U^n - u_h(t_n))\big\| = \|G_h F_n v\| \le C_0 \big(h^{r+2}|v|_r + k^p |v|_{2p}\big).$$

By Thomée [21, Theorem 7.8], we have

$$\|U^n - u_h(t_n)\| \le C_0 \big(h^r |v|_r + k^p |v|_{2p}\big), \quad t_n \ge 0.$$

Thus,

$$\begin{aligned}\big|U^n - u_h(t_n)\big|_{-2} &\le \big\|(G-G_h)(U^n - u_h(t_n))\big\| + \big\|G_h(U^n - u_h(t_n))\big\| \\ &\le C_0\big(h^{r+2}|v|_r + k^p|v|_{2p}\big).\end{aligned}$$

Together these estimates complete the proof. □

Now we turn to the proof of Lemma 6.3. The idea of the proof is similar to the one used in Lemma 6.4

Proof of Lemma 6.3. By Thomée [21, Theorem 6.4], we have

$$\big|u_{h,t}(t) - u_t(t)\big|_{-2} \le C h^{r+2} |v|_{r+2}.$$

Therefore it suffices to show

$$\big|Q_k U^n - u_{h,t}(t_n)\big|_{-2} \le C\big(h^{r+2}|v|_{r+2} + k^p |v|_{2(p+1)}\big), \tag{6.14}$$

which we will prove now.

We first estimate $\big|Q_k U^n - u_{h,t}(t_n)\big|_{-2,h}$. Noting that, with $v_h = R_h v = G_h A v$,

$$\begin{aligned}Q_k U^n - u_{h,t}(t_n) &= k^{-1}\Big(\sum_{\nu=-m_1}^{m_2} c_\nu U^{n+\nu} - (-A_h)e^{-nkA_h}\Big) G_h A v \\ &= k^{-1} g_n(kA_h) G_h A v,\end{aligned}$$

where $g_n(\lambda) = \sum_{\nu=-m_1}^{m_2} r(\lambda)^{n+\nu} - (-\lambda)e^{-n\lambda}$, we need to show

$$\big\|G_h\big(k^{-1} g_n(kA_h) G_h A v\big)\big\| \le C_0\big(h^{r+2}|v|_{r+2} + k^p |v|_{2(p+1)}\big).$$

As in the proof of Lemma 6.4, we introduce v_k which satisfies:

$$\|A(v - v_k)\| \le k^p |v|_{2p+2}, \tag{6.15}$$

$$|v_k|_{2(p+1)} \le |v|_{2(p+1)}, \tag{6.16}$$

and

$$|v_k|_{r+2l+2} \le k^{-l}|v|_{r+2}, \quad \text{for } 0 \le l \le p-1. \tag{6.17}$$

Applying now the identity (6.11) to $v = Av_k$, we get

$$\begin{aligned} G_h g_n(kA_h) G_h A v_k = \sum_{l=0}^{p-1} & g_n(kA_h) G_h^{l+1}\big(G_h(G-G_h)A^{l+2}v_k\big) \\ &+ G_h g_n(kA_h) G_h^{p+1} A^{p+1} v_k. \end{aligned}$$

It is easy to show that, see, e.g., [24, Lemma 3.9],

$$\|g_n(kA_h)G_h^{l+1}\| \le C_0 k^{l+1}, \quad \text{for } 0 \le l \le p,\ n \ge 0. \tag{6.18}$$

Thus, by (6.16) and noting the boundedness of G_h,

$$\begin{aligned} \|G_h g_n(kA_h) G_h^{p+1} A^{p+1} v_k\| &\le \|g_n(kA_h) G_h^{p+1} A^{p+1} v_k\| \\ &\le C_0 k^{p+1}\|A^{p+1}v_k\| \le C_0 k^{p+1}|v_k|_{2(p+1)} \le C_0 k^{p+1}|v|_{2(p+1)}. \end{aligned}$$

Further, by (6.17), (6.18), and using (2.11), we have, with $0 \le l \le p-1$,

$$\begin{aligned} \big\|g_n(kA_h)G_h^{l+1}\big(G_h(G-G_h)A^{l+2}v_k\big)\big\| &\le C_0 k^{l+1}\|G_h(G-G_h)A^{l+2}v_k\| \\ \le C_0 k^{l+1}h^2\|(G-G_h)(A^{l+2}v_k)\| &+ C_0 k^{l+1}h^{r+2}|A^{l+2}v_k|_{r-2} \\ \le C_0 k^{l+1}h^{r+2}\|A^{l+2}v_k\|_{r-2} &\le C_0 k^{l+1}h^{r+2}|v_k|_{r+2l+2} \le C_0 k h^{r+2}|v|_{r+2}. \end{aligned}$$

Together these estimates imply

$$\|G_h g_n(kA_h) G_h A v_k\| \le C_0 k(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}).$$

Since obviously, by (6.15), the boundedness of G_h and stability, we get

$$\begin{aligned} \|G_h g_n(kA_h) G_h A(v-v_k)\| &\le \|g_n(kA_h) G_h A(v-v_k)\| \\ &\le C_0 k\|A(v-v_k)\| \le C_0 k^{p+1}|v|_{2(p+1)}, \end{aligned}$$

we conclude that

$$\begin{aligned} \|G_h(Q_k U^n - u_{h,t}(t_n))\| &= k^{-1}\|G_h g_n(kA_h) G_h A v\| \\ &\le C_0(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}). \end{aligned}$$

By [24, Theorem 3.8], we have

$$\|Q_k U^n - u_{h,t}(t_n)\| \le C_0(h^r|v|_{r+2} + k^p|v|_{2p}).$$

Thus

$$\begin{aligned} |Q_k U^n - u_{h,t}(t_n)|_{-2} &\le \|(G-G_h)(Q_k U^n - u_{h,t}(t_n))\| \\ &\quad + \|G_h(Q_k U^n - u_{h,t}(t_n))\| \\ &\le C_0(h^{r+2}|v|_{r+2} + k^p|v|_{2(p+1)}). \end{aligned}$$

Together these estimates complete the proof. □

After the preparations above we now come to the proof of Theorem 6.1.

Proof of Theorem 6.1. Combining (6.6) and (6.1), we get, with $\tilde{G}_h = \tilde{A}_h^{-1}$,

$$\begin{aligned}\tilde{u}_h(T) - u(T) &= \tilde{G}_h \tilde{P}_h(-Q_k U^n) - G(-u_t) \\ &= (\tilde{G}_h \tilde{P}_h - G)(-Q_k U^n + u_t(t_n)) \\ &\quad - (\tilde{G}_h \tilde{P}_h - G) u_t(t_n) + G(Q_k U^n - u_t).\end{aligned}$$

Thus, by Lemmas 6.2 and 6.3, and noting that $\|(\tilde{G}_h \tilde{P}_h - G) f\| \le C h^s \|f\|_{s-2}$, for $0 \le s \le r+2$, we have

$$\begin{aligned}\|\tilde{u}_h(T) - u(T)\| &\le C_0 h^2 \|Q_k U^n - u_t(t_n)\| \\ &\quad + C_0 h^{r+2} \|u_t(t_n)\|_r + |(Q_k U^n - u_t(t_n))|_{-2} \\ &\le C_0 \big(h^{r+2} |v|_{r+2} + k^p |v|_{2(p+1)} + h^{r+2} \|u_t(t_n)\|_r \big).\end{aligned}$$

Together these estimates complete the proof. □

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References

1. G. Akrivis, M. Crouzeix, and C. Makridakis, *Implicit-explicit multistep finite element methods for nonlinear parabolic problems*, Math. Comp. **67** (1998), 457–477.
2. ______, *Implicit-explicit multistep methods for quasilinear parabolic equations*, Numer. Math. **82** (1999), 521–541.
3. S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.
4. C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer Series in Computational Physics, Springer-Verlag, New York, 1988.
5. M. Crouzeix, V. Thomée, and L. B. Wahlbin, *Error estimates for spatially discrete approximations of semilinear parabolic equations with initial data of low regularity*, Math. Comp. **53** (1989), 25–41.
6. J. de Frutos, B. García-Archilla, and J. Novo, *A postprocessed Galerkin method with Chebyshev or Legendre polynomials*, Numer. Math. **86** (2000), 419–442.
7. J. de Frutos and J. Novo, *A postprocess based improvement of the spectral element method*, Appl. Numer. Math. **33** (2000), 217–223.
8. ______, *Postprocessing the linear finite element method*, SIAM J. Numer. Anal. **40** (2000), 805–819.
9. ______, *A spectral element method for the Navier-Stokes equations with improved accuracy*, SIAM J. Numer. Anal. **38** (2000), 799–819.
10. C. M. Elliott and S. Larsson, *Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation*, Math. Comp. **58** (1992), 603–630.
11. ______, *A finite element model for the time-dependent Joule heating problem*, Math. Comp. **64** (1995), 1433–1453.
12. C. M. Elliott and A. M. Stuart, *The global dynamics of discrete semilinear parabolic equations*, SIAM J. Numer. Anal. **30** (1993), 1622–1663.
13. B. García-Archilla and E. S. Titi, *Postprocessing the Galerkin method: the finite-element case*, SIAM J. Numer. Anal. **37** (2000), 470–499.
14. H. P. Helfrich, *Error estimates for semidiscrete Galerkin type approximations to semilinear evolution equations with nonsmooth initial data*, Numer. Math. **51** (1987), 559–569.

15. C. Johnson, S. Larsson, V. Thomée, and L. B. Wahlbin, *Error estimates for spatially discrete approximations of semilinear parabolic equations with nonsmooth initial data*, Math. Comp. **49** (1987), 331–357.
16. S. Larsson, *Nonsmooth data error estimates with applications to the study of the long-time behavior of finite element solutions of semilinear parabolic problems*, Preprint 1992–36, Department of Mathematics, Chalmers University of Technology.
17. ———, *The long-time behavior of finite-element approximations of solutions to semilinear parabolic problems*, SIAM J. Numer. Anal. **26** (1989), 348–365.
18. S. Larsson and J.-M. Sanz-Serna, *The behavior of finite element solutions of semilinear parabolic problems near stationary points*, SIAM J. Numer. Anal. **31** (1994), 1000–1018.
19. ———, *A shadowing result with applications to finite element approximation of reaction-diffusion equations*, Math. Comp. **68** (1999), 55–72.
20. A. H. Schatz, V. Thomée, and L. B. Wahlbin, *Stability, analyticity, and almost best approximation in maximum norm for parabolic finite element equations*, Comm. Pure Appl. Math. **51** (1998), 1349–1385.
21. V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, Berlin, 1997.
22. V. Thomée and L. Wahlbin, *On Galerkin methods in semilinear parabolic problems*, SIAM J. Numer. Anal. **12** (1975), 378–389.
23. M. F. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal. **10** (1973), 723–759.
24. Y. Yan, *Approximation of time derivatives for parabolic equations in Banach space: constant time steps*, Preprint 2002–01, Chalmers Finite Element Center, Chalmers University of Technology, to appear in IMA J. Numer. Anal.

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