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A LAGRANGE MULTIPLIER METHOD FOR THE FINITE ELEMENT SOLUTION OF ELLIPTIC DOMAIN DECOMPOSITION PROBLEMS USING NON-MATCHING MESHES

PETER HANSBO, CARLO LOVADINA, ILARIA PERUGIA, AND GIANCARLO SANGALLI

Abstract. In this paper we propose a Lagrange multiplier method for the finite element solution of elliptic partial differential equations using domain decomposition with non-matching meshes. The interface Lagrange multiplier is discretized by means of global polynomials, in order to avoid the cumbersome integration of products of unrelated mesh functions. The ideas are illustrated using Poisson's equation as a model, and the proposed method is shown to be stable and optimally convergent. Numerical experiments demonstrating the theoretical results are also presented.

1. Introduction

When considering domain decomposition with non-matching meshes using Lagrange multiplier techniques, two basic problems occur. First and foremost, the relation between the discrete spaces chosen for the primal variable and the multipliers must be such that the resulting numerical scheme is stable. Proving stability reduces to proving that the approximate solution fulfills the *inf-sup* condition [5]; it then turns out that many natural choices of approximations do not. Fortunately, this problem can be alleviated by use of stabilized multiplier methods [9, 11, 1, 4], or by using mesh-dependent penalty methods [2, 3]. The second problem is that products of traces of the primal variable and the multipliers have to be integrated on the interfaces. For methods known to fulfill the *inf-sup* condition, such as the mortar element method (see [12] for an overview of such methods), as well as most stabilized methods, this will mean integrating products of piecewise polynomials on unrelated meshes. This is not easily done in practice for problems in $\mathbb{R}^3$ (see, however, [10]). To mitigate these two problems, we suggest a stabilization method which allows for global polynomial multipliers on the interfaces. This method is stable and optimally

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convergent, and, moreover, avoids the cumbersome integration of products of unrelated mesh functions. Only products of global polynomials and local polynomials have to be considered; this makes the integration problem much simpler in many cases. The method is presented and analyzed using a two-dimensional Poisson equation as a model with two subdomains. Form a geometric point of view, two situations can occur: the case where the intersections of the boundaries of both subdomains with the outer boundary have non-zero 1D measure, and the case where the interface is a closed curve (see Fig. 1 below). We point out that the stability analysis of the former case, for which similar arguments as in [1] could be applied, is simpler than that of the latter case. In this paper we present an analysis that covers both cases.

As a basic motivation for this work, we have in mind the contact problem in elasticity. In standard commercial codes for computing contact between two elastic bodies, the contact condition is only checked at the nodes either on one or on both of the bodies. This corresponds to choosing discrete Lagrange multipliers which is not natural from the perspective of the variational formulation of the problem. The stability and convergence properties of these approaches are in general not known, and the results have to be carefully interpreted, which requires some experience. Furthermore, in our experience, plenty of choices have to be made in discretizing the interface, choosing the master surface, etc. An obvious reason for choosing discrete multipliers is that it makes the integration problem particularly easy; as mentioned above this is also the aim of the method to be presented.

One could interpret our approach as covering the contact surface with a polynomial layer which acts as an intermediate between the two surfaces (which do not have to be known in advance). Even though this results in a global coupling of all the variables on the contact surface, the typical contact application has a small zone of contact and the global coupling will not cause the problem to grow excessively in size.

The outline of the paper is as follows. In Section 2 the interface Lagrange multiplier method with global polynomial discretization of the multiplier is presented, after introducing the model problem together with some notation, and discussing the motivation of the present work. The stability and error analysis of the new method is carried out in Section 3, and numerical experiments demonstrating the theoretical results are presented in Section 4. The paper ends with some conclusions in Section 5.

2. Formulation of the method

In this section we introduce a novel interface Lagrange multiplier method for the finite element discretization of elliptic problems on non-matching grids. Before doing that, we make precise the model problem we will be working on, together with some notation and motivation of the present work.

2.1. **Model problem.** Let Ω be a bounded domain in $\mathbb{R}^2$, with boundary $\partial\Omega$. (The extension to $\mathbb{R}^3$ is straightforward.) As a model problem, we consider a stationary heat conduction problem in the case where there is a piecewise smooth internal boundary Γ

dividing Ω into two subdomains Ω_1 and Ω_2. Thus, we want to solve for u the problem

$$
\begin{aligned}
-\nabla\cdot(\kappa_i \nabla u_i) &= f \quad \text{in} \quad \Omega_i,\\
u_i &= 0 \quad \text{on} \quad \partial\Omega_i \cap \partial\Omega,\\
u_1 - u_2 &= 0 \quad \text{on} \quad \Gamma,\\
\boldsymbol{n}_1\cdot\kappa_1\nabla u_1 + \boldsymbol{n}_2\cdot\kappa_2\nabla u_2 &= 0 \quad \text{on} \quad \Gamma,
\end{aligned}
\tag{2.1}
$$

for $i = 1, 2$, where we have denoted by u_i the restriction of u to Ω_i. Here f is a given function, κ_i, which is assumed to be smooth in Ω_i, is the conductivity, and $\boldsymbol{n}_i$ is the outward pointing normal to Ω_i at Γ, $i = 1, 2$. This formulation of the standard Poisson problem is common in the context of domain decomposition problems, cf. [7].

Define

$$V = \{v : v_i \in H^1(\Omega_i),\ v_i = 0 \text{ on } \partial\Omega_i \setminus \Gamma,\ i = 1, 2\}$$

and

$$\Lambda = \left(H^{1/2}_{00}(\Gamma)\right)',$$

the dual space of $H^{1/2}_{00}(\Gamma)$ (see, e.g., [8]). Notice that, whenever $\Gamma \cap \partial\Omega = \emptyset$ (see Fig. 1, right), $H^{1/2}_{00}(\Gamma)$ coincides with $H^{1/2}(\Gamma)$ and $\Lambda = H^{-1/2}(\Gamma)$.

A weak form of (2.1) using the Lagrange multiplier approach is as follows:
Find $(u, \lambda) \in V \times \Lambda$ such that

$$
\begin{aligned}
&\sum_i \int_{\Omega_i} \kappa_i \nabla u_i \cdot \nabla v_i \, dx + \int_\Gamma \lambda\, [v]\, ds = \sum_i \int_{\Omega_i} f\, v_i \, dx \quad \forall v \in V,\\
&\int_\Gamma [u]\, \mu \, ds = 0 \quad \forall \mu \in \Lambda,
\end{aligned}
\tag{2.2}
$$

where $[v] := (v_1 - v_2)|_\Gamma$ is the jump of v across Γ. Notice that

$$\lambda = -\kappa_1 \nabla u_1 \cdot \boldsymbol{n}_1 = \kappa_2 \nabla u_2 \cdot \boldsymbol{n}_2 \quad \text{on } \Gamma.$$

2.2. **Notation.** We introduce the necessary notation for the definition of the method we are going to present and its subsequent analysis, focusing, for simplicity, on the case of triangular elements. Therefore, we assume that we are given a triangular mesh $\mathcal{T}_i^h$ of the domain Ω_i, $i = 1, 2$. We denote by h_i the meshsize of $\mathcal{T}_i^h$. Obviously, $\mathcal{T}^h = \mathcal{T}_1^h \cup \mathcal{T}_2^h$ provides a mesh for Ω, whose meshsize is $h = \max\{h_1, h_2\}$. We introduce the finite element space

$$V^h = \{v \in V :\ v|_K \in P^k(K),\ \forall K \in \mathcal{T}^h\},$$

where $P^k(K)$ denotes the space of polynomials of degree at most k on K, with $k \geq 1$.

The interface Γ is decomposed as the union $\Gamma = \bigcup \Gamma_j$ of n_Γ smooth components Γ_j of length ℓ_j (see Fig. 1). We associate with each Γ_j the non-negative integer p_j and define for later use $\underline{p} := [p_1, \ldots, p_{n_\Gamma}]$. On Γ, we introduce the space

$$\Lambda^p = \{\mu \in \Lambda :\ \mu|_{\Gamma_j} \in P^{p_j}(\Gamma_j),\ j = 1, \ldots, n_\Gamma\},$$

where $P^{p_j}(\Gamma_j)$ denotes the space of polynomials of degree at most p_j on Γ_j, with respect to a local coordinate. Notice that the elements of Λ^p can be discontinuous at the endpoints of the Γ_j's. We also remark that two different situations can occur from a geometric point of view:

(1) both $\partial\Omega_1 \cap \partial\Omega$ and $\partial\Omega_2 \cap \partial\Omega$ have nonzero 1-D measure (see Fig. 1, left);
(2) either $\partial\Omega_1 \cap \partial\Omega$ or $\partial\Omega_2 \cap \partial\Omega$ has zero 1-D measure (see Fig. 1, right).

As we will see in the next section, the stability analysis is more difficult for the second case.

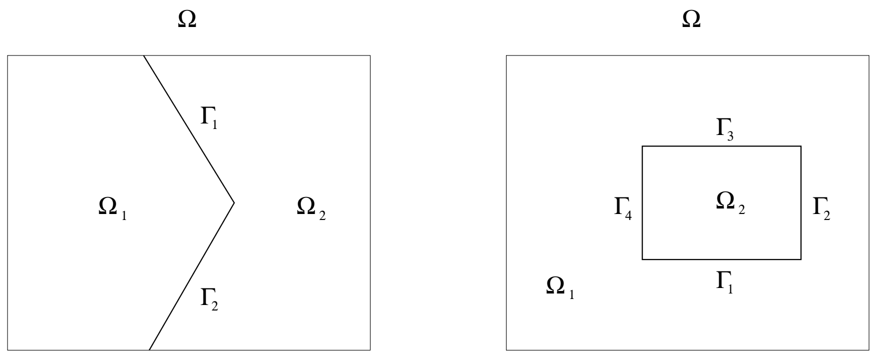

FIGURE 1. Two different geometric situations.

2.3. **Background.** A standard penalty method for domain decomposition problems is the following (cf. [2]):
Find $u^h \in V^h$ such that

$$\sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \gamma \, [u^h] \, [v] \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \qquad \forall v \in V^h. \tag{2.3}$$

There is a consistency error present in (2.3), but by letting γ depend inversely on the meshsize, i.e., $\gamma = Ch^{-\alpha}$, for suitable values of α this consistency error will not dominate the discretization error in energy-like norms (see [3] for an extensive investigation).

The main problem of implementation of (2.3) is how to evaluate integrals of the type

$$\int_\Gamma u_i^h \, v_j \, ds, \quad i \neq j,$$

especially in three dimensions. If quadrature is used, we have an expensive search problem in locating elements in the mesh on Ω_i containing quadrature points in the elements of the mesh on Ω_j. Exact integration is within reach; Priestley [10] has made an implementation of exact quadrature in the case of triangular surface meshes with common boundary. Here we take an alternative route which greatly simplifies the implementation, by introducing a suitable multiplier on the interface Γ.

Therefore, starting from (2.3), one could consider the following *inconsistent* perturbed Lagrange multiplier method:

Find $(u^h, \lambda^p) \in V^h \times \Lambda^p$ such that

$$\begin{aligned} &\sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \lambda^p \, [v] \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \quad \forall v \in V^h, \\ &\int_\Gamma [u^h] \, \mu \, ds - \int_\Gamma \frac{1}{\gamma} \, \lambda^p \, \mu \, ds = 0 \quad \forall \mu \in \Lambda^p. \end{aligned} \tag{2.4}$$

Now, as $\gamma \to \infty$ the problem will be a standard saddle-point problem which requires balancing between the discrete spaces for λ^p and u^h which have to fulfill a Babuška-Brezzi condition [5]. Again, we can let $\gamma = Ch^{-\alpha}$, so that the problem of balancing Λ^p and V^h is alleviated. We can thus freely use the rule of thumb that the number of degrees of freedom of Λ^p should approximately match the number of degrees of freedom on Γ from the meshes adjacent to it. Furthermore, the product of basis functions and global polynomials can easily be integrated exactly, at least for simplicial elements.

On the other hand, due to inconsistency, the formulation (2.4) exhibits a loss of accuracy, as our numerical results in Section 4 indicate. It also requires coupling the parameter α to the polynomial degree of approximation, which is detrimental to the conditioning of the system. In order to overcome these drawbacks, we are going to present a *consistent* version of (2.4).

2.4. **The consistent method: A Nitsche-type interface condition.** The classical method of Nitsche [9] for handling Dirichlet boundary conditions weakly was extended to the case of domain decomposition with non-matching meshes by Becker, Hansbo and Stenberg [4]. The problem of having to integrate products of functions on one side of the interface with functions on the other side is present also in their method. However, Nitsche-type methods are consistent and thus optimally convergent with a fixed "penalty" parameter of $O(h^{-1})$, which does not destroy the conditioning of the resulting system. Thus, there could be some advantages to formulating a Nitsche-type interface formulation using the interjacent polynomial multiplier space. To this end we define $\boldsymbol{n} := \boldsymbol{n}_1 = -\boldsymbol{n}_2$ on Γ and

$$\{u\} := \alpha u_1 + (1 - \alpha) u_2,$$

where $0 \leq \alpha \leq 1$, and propose the following method:
Find $(u^h, \lambda^p) \in V^h \times \Lambda^p$ such that

$$\begin{aligned} &\sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \lambda^p \, [v] \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \quad \forall v \in V^h, \\ &\int_\Gamma [u^h] \, \mu \, ds - \int_\Gamma \frac{1}{\gamma} \left\{ \boldsymbol{n} \cdot \kappa \nabla u^h \right\} \mu \, ds - \int_\Gamma \frac{1}{\gamma} \, \lambda^p \, \mu \, ds = 0 \quad \forall \mu \in \Lambda^p. \end{aligned} \tag{2.5}$$

We note that (2.5) is a *consistent* method: inserting a sufficiently regular analytical solution (u, λ) in the place of (u^h, λ^p), since formally $\lambda = -\kappa_1 \nabla u_1 \cdot \boldsymbol{n}_1 = \kappa_2 \nabla u_2 \cdot \boldsymbol{n}_2$ on Γ, we find that

$$\sum_i \int_{\Omega_i} \kappa_i \nabla(u_i - u_i^h) \cdot \nabla v_i \, dx + \int_\Gamma (\lambda - \lambda^p) \, [v] \, ds = 0,$$
$$\int_\Gamma [u - u^h] \, \mu \, ds - \int_\Gamma \frac{1}{\gamma} \left\{ \boldsymbol{n} \cdot \kappa \nabla(u - u^h) \right\} \mu \, ds - \int_\Gamma \frac{1}{\gamma} \, (\lambda - \lambda^p) \, \mu \, ds = 0,$$

for all $v \in V^h$ and $\mu \in \Lambda^p$. We rephrase this property in abstract form in the following lemma, where we set

$$\begin{aligned} \mathcal{B}^h(w, \nu; v, \mu) \quad := \quad & \sum_i \int_{\Omega_i} \kappa_i \nabla w_i \cdot \nabla v_i \, dx + \int_\Gamma \nu \, [v] \, ds - \int_\Gamma [w] \, \mu \, ds \\ & + \int_\Gamma \frac{1}{\gamma} \, \{\boldsymbol{n} \cdot \kappa \nabla w\} \, \mu \, ds + \int_\Gamma \frac{1}{\gamma} \, \nu \, \mu \, ds. \end{aligned}$$

Lemma 1. *The method* (2.5) *is consistent in the sense that*

$$\mathcal{B}^h(u - u^h, \lambda - \lambda^p; v, \mu) = 0,$$

for all $v \in V^h$ *and* $\mu \in \Lambda^p$.

Remark 1. *A drawback of the method* (2.5) *is that it is* unsymmetric*. However, in the cases of* $\alpha = 0$ *and* $\alpha = 1$*, it can be symmetrized without introducing integration of cross terms across the interface and without altering the consistency. The symmetric and consistent formulation reads as follows:*

$$\begin{aligned} & \sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \lambda^p \, [v] \, ds - \int_\Gamma \frac{1}{\gamma} \lambda^p (\boldsymbol{n}_j \cdot \kappa_j \nabla v_j) \, ds \\ & \qquad - \int_\Gamma \frac{1}{\gamma} (\boldsymbol{n}_j \cdot \kappa_j \nabla u_j^h)(\boldsymbol{n}_j \cdot \kappa_j \nabla v_j) \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \quad \forall v \in V^h, \\ & \int_\Gamma [u^h] \, \mu \, ds - \int_\Gamma \frac{1}{\gamma} (\boldsymbol{n}_j \cdot \kappa_j \nabla u_j^h) \mu \, ds - \int_\Gamma \frac{1}{\gamma} \, \lambda^p \, \mu \, ds = 0 \quad \forall \mu \in \Lambda^p, \end{aligned}$$

with subscript $j = 2$ *if* $\alpha = 0$*, and* $j = 1$ *if* $\alpha = 1$*. We remark that, as pointed out in the discussion after Theorem 1 below,* $\alpha = 0$ *and* $\alpha = 1$ *are reasonable choices and in particular the sole choices which gives the best convergence results whenever the characteristic mesh-sizes of the two domains are significantly different from each other. An additional reason for using the symmetric formulation is its adjoint consistency, which is useful whenever duality arguments need to be applied.*

3. Analysis of the method

In this section we develop stability analysis and derive error estimates of method (2.5). The analysis will show that, in the particular case of quasi-uniform meshes $\mathcal{T}^h$ of size h, the choice of the polynomial approximation orders for λ that give the best error estimate is $p_j \approx \ell_j/h$, $j = 1, \ldots, n_\Gamma$. In correspondence to that, and for γ of the order of h^{-1}, for analytical solutions in $H^{s+1}(\Omega_i)$ for each subdomain Ω_i, we obtain the optimal error estimate

$$|||(u-u^h,\lambda-\lambda^p)||| \leq Ch^{\min\{k,s\}}\sum_i |u_i|_{H^{s+1}(\Omega_i)},$$

where $|||\cdot|||$ is an energy-like norm, and C is a positive constant independent of h. More in general, the estimates we will derive take into account the *local* meshsize and material properties. This provides criteria on how to select the parameter γ and the polynomial approximation degrees p_j, $j=1,\dots,n_\Gamma$, in the most advantageous way, also in the case of meshes of different sizes in the different subdomains Ω_i.

3.1. **Stability.** Defining the (weighted) broken H^1–norm

$$\|w\|_V = \Big(\sum_i \|\kappa_i^{1/2}\nabla w_i\|^2_{L_2(\Omega_i)} + \|aw\|^2_{L_2(\Omega)}\Big)^{1/2},$$

with $a=\kappa^{1/2}/\mathrm{diam}(\Omega)$, we introduce the norm

$$|||(w,\nu)||| := \big(\|w\|_V^2 + \|\gamma^{-1/2}\nu\|^2_{L_2(\Gamma)}\big)^{1/2},$$

where γ is the function of $L^\infty(\Gamma)$ defined as follows. Denote by $\mathcal{N}_\Gamma$ the set of nodes of $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ lying on Γ. Fix a point $\boldsymbol{x}$ on $\Gamma\setminus\mathcal{N}_\Gamma$ and let K_1 and K_2 be the two elements of $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$, respectively, such that the interior of $\partial K_1\cap\partial K_2\cap\Gamma$ is non empty and $\boldsymbol{x}\in\partial K_1\cap\partial K_2\cap\Gamma$. Denote by h_{K_1} and h_{K_2} the diameters of K_1 and K_2, respectively. For $\boldsymbol{x}\in\Gamma\setminus\mathcal{N}_\Gamma$, we define

$$\gamma(\boldsymbol{x}) = \gamma_0\,\max\{\alpha\kappa_1(\boldsymbol{x})h_{K_1}^{-1},(1-\alpha)\kappa_2(\boldsymbol{x})h_{K_2}^{-1}\},$$

with γ_0 constant independent on the meshsize and the material properties. Restrictions on γ_0 will be made precise later on. Notice that γ is defined almost everywhere on Γ.

Define the constant $\mathtt{k}$ as the mean value of the function κ on Ω and $b:=\big(\mathtt{k}/\mathrm{diam}(\Omega)\big)^{1/2}$. On the interface Γ, we will use the norm

$$[\![\varphi]\!]_{1/2,\Gamma} := \Big(\|b\varphi\|^2_{L_2(\Gamma)} + |\mathtt{k}^{1/2}\varphi|^2_{H^{1/2}_{00}(\Gamma)}\Big)^{1/2},$$

together with its dual denoted by $[\![\cdot]\!]_{-1/2,\Gamma}$. We recall that, whenever $\Gamma\cap\partial\Omega=\emptyset$, $|\cdot|_{H^{1/2}_{00}(\Gamma)}$ coincides with $|\cdot|_{H^{1/2}(\Gamma)}$ (see [8]).

Remark 2. *The norm $[\![\cdot]\!]_{1/2,\Gamma}$ is the natural norm for the traces on Γ of functions belonging to V, when V is endowed with the $\|\cdot\|_V$–norm.*

We proceed by proving continuity and inf-sup properties of the form $\mathcal{B}^h$. In the sequel C, $C_1,\dots$ denote generic *strictly positive* constants possibly depending on the shape of the domain, on the shape regularity of the meshes, on the quantity $\max\kappa/\min\kappa$, and on the polynomial approximation degrees of V^h, but independent of the meshsize and of $\underline{p}$.

Proposition 1. *For all w, $v\in V$, ν, $\mu\in\Lambda$ we have*

$$\mathcal{B}^h(w,\nu;v,\mu) \leq C\left(|||(w,\nu)||| + \|\gamma^{1/2}[w]\|_{L_2(\Gamma)} + [\![\nu]\!]_{-1/2,\Gamma}\right)|||(v,\mu)|||. \tag{3.1}$$

Proof. The bound follows from the Cauchy-Schwarz inequality and from

$$[\![\,[w]\,]\!]_{1/2,\Gamma} \leq [\![w_1]\!]_{1/2,\Gamma} + [\![w_2]\!]_{1/2,\Gamma} \leq C\|w\|_V$$

(see Remark 1). □

Proposition 2. *Provided that γ_0 is large enough (see Remark 2 below), for all $(w,\nu) \in V^h \times \Lambda^p$ there is $(v,\mu) \in V^h \times \Lambda^p$ such that*

$$\begin{aligned} &|\!|\!|(v,\mu)|\!|\!| \leq C_1 |\!|\!|(w,\nu)|\!|\!|, \\ &\mathcal{B}^h(w,\nu;v,\mu) \geq C_2 |\!|\!|(w,\nu)|\!|\!|^2, \end{aligned}$$

Proof. Define $P^0(\Gamma)$ as the space of constants on the whole interface Γ. Let $(w,\nu) \in V^h \times \Lambda^p$ and take $(v,\mu) \in V^h \times \Lambda^p$ as $v = w$ and $\mu = \mu_1 + \delta\mu_2$, with $\mu_1 = \nu$ and $\mu_2 = -b^2\Pi_0[w]$, where Π_0 is the $L_2(\Gamma)$–projection operator onto $P^0(\Gamma)$, and δ is a positive parameter still at our disposal. From the definition of μ_2, the bound $b\gamma^{-1/2} \leq C{\gamma_0}^{-1/2}$, and a trace inequality, we have

$$\|\gamma^{-1/2}\mu_2\|_{L_2(\Gamma)} = b\|(b\gamma^{-1/2})\Pi_0[w]\|_{L_2(\Gamma)} \leq C{\gamma_0}^{-1/2} b\|[w]\|_{L_2(\Gamma)} \leq C{\gamma_0}^{-1/2}\|w\|_V.$$

The continuity estimate $|\!|\!|(v,\mu)|\!|\!| \leq C_1|\!|\!|(w,\nu)|\!|\!|$ immediately follows.

For the coercivity, we proceed in two steps. First, from the definition of v and μ_1 we have

$$\mathcal{B}^h(w,\nu;v,\mu_1) = \sum_i \|\kappa_i^{1/2}\nabla w_i\|^2_{L_2(\Omega_i)} + \int_\Gamma \frac{1}{\gamma}\{\boldsymbol{n}\cdot\kappa\nabla w\}\,\nu\,ds + \|\gamma^{-1/2}\nu\|^2_{L_2(\Gamma)}.$$

By combining a weighted Cauchy-Schwarz inequality with the inverse inequality

$$\|\kappa_i^{-1/2}h_i^{1/2}\boldsymbol{n}_i\cdot\kappa_i\nabla w_i\|^2_{L_2(\Gamma)} \leq C_I\|\kappa_i^{1/2}\nabla w_i\|^2_{L_2(\Omega_i)}, \qquad \forall w \in V_i^h,\ i = 1,2, \tag{3.2}$$

provided that $\gamma_0 > C_I/4$, making use of the Young inequality, we obtain

$$\mathcal{B}^h(w,\nu;v,\mu_1) \geq C\Big(\sum_i \|\kappa_i^{1/2}\nabla w_i\|^2_{L_2(\Omega_i)} + \|\gamma^{-1/2}\nu\|^2_{L_2(\Gamma)}\Big), \tag{3.3}$$

where C depends on C_I and γ_0, but is independent of the meshsize.

For the second step, we can write

$$\begin{aligned} \mathcal{B}^h(w,\nu;0,\mu_2) &= \int_\Gamma b\,[w]\,b\,\Pi_0[w]\,ds - \int_\Gamma b^2\gamma^{-1}\{\boldsymbol{n}\cdot\kappa\nabla w\}\,\Pi_0[w]\,ds - \int_\Gamma b^2\gamma^{-1}\,\nu\,\Pi_0[w]\,ds \\ &= \|b\,\Pi_0[w]\|^2_{L_2(\Gamma)} - \int_\Gamma b^2\gamma^{-1}\{\boldsymbol{n}\cdot\kappa\nabla w\}\,\Pi_0[w]\,ds - \int_\Gamma b^2\gamma^{-1}\,\nu\,\Pi_0[w]\,ds. \end{aligned}$$

Using suitably weighted Cauchy-Schwarz inequalities for the last two integrals and the inverse inequality (3.2), we obtain

$$\mathcal{B}^h(w,\nu;0,\mu_2) \geq \frac{1}{2}\|b\,\Pi_0[w]\|^2_{L_2(\Gamma)} - \frac{CC_I}{\gamma_0^2}\sum_i \|\kappa_i^{1/2}\nabla w_i\|^2_{L_2(\Omega_i)} - \frac{C}{\gamma_0}\|\gamma^{-1/2}\nu\|^2_{L_2(\Gamma)}. \tag{3.4}$$

By adding together (3.3) and (3.4) multiplied by δ, and taking δ small enough (depending on the constants C, C_I and γ_0), we obtain

$$\mathcal{B}^h(w,\nu;v,\mu) \geq C_3 \left(\sum_i \|\kappa_i^{1/2}\nabla w_i\|^2_{L_2(\Omega_i)} + \|b\,\Pi_0[w]\|^2_{L_2(\Gamma)} + \|\gamma^{-1/2}\nu\|^2_{L_2(\Gamma)} \right), \tag{3.5}$$

with a positive constant C_3 only depending on C_I and γ_0, therefore independent of the meshsize.

In order to complete the proof of the proposition, we need to show that

$$\|aw\|^2_{L_2(\Omega)} \leq C \left(\sum_i \|\kappa_i^{1/2}\nabla w_i\|^2_{L_2(\Omega_i)} + \|b\,\Pi_0[w]\|^2_{L_2(\Gamma)} \right), \tag{3.6}$$

with a constant C independent of the meshsize. Then, the coercivity estimate $\mathcal{B}^h(w,\nu;v,\mu) \geq C_2|||(w,\nu)|||^2$ easily follows. In the case where both $\partial\Omega_1$ and $\partial\Omega_2$ contain a part of the Dirichlet boundary with nonzero 1–dimensional measure (see Fig. 1, left), the estimate (3.6) directly follows from the standard Poincaré's inequality. On the other hand, if either of $\partial\Omega_1$ or $\partial\Omega_2$ does not contain a part of the Dirichlet boundary with nonzero 1–dimensional measure (see Fig. 1, right), we still make use of the Poincaré inequality, but the proof is not straightforward. We develop this case in detail.

Assume, to fix the ideas, that $\partial\Omega_2$ does not intersect $\partial\Omega$. Then, $\partial\Omega_1$ contains the Dirichlet boundary. Therefore, the following Poincaré's inequality holds true:

$$\|w_1\|_{L_2(\Omega_1)} \leq C\,\mathrm{diam}(\Omega_1)\,\|\nabla w_1\|_{L_2(\Omega_1)}. \tag{3.7}$$

Write w_2 as $(w_2 - \Pi_0 w_{2|_\Gamma}) + \Pi_0 w_{2|_\Gamma}$, where $\Pi_0 w_{2|_\Gamma}$ is the constant function on Ω_2 equal to the mean value of the trace of w_2 on Γ. The Poincaré's inequality

$$\|w_2 - \Pi_0 w_{2|_\Gamma}\|_{L_2(\Omega_2)} \leq C\,\mathrm{diam}(\Omega_2)\,\|\nabla w_2\|_{L_2(\Omega_2)}$$

follows from Bramble-Hilbert's lemma applied to the operator $\pi : H^1(\Omega_2) \to L_2(\Omega_2)$ defined by $\pi(w) = w - \Pi_0 w_{|_\Gamma}$, which is zero on $P^0(\Omega_2)$, the space of constants on Ω_2. Therefore,

$$\begin{aligned}
\|w_2\|_{L_2(\Omega_2)} &\leq \|w_2 - \Pi_0 w_{2|_\Gamma}\|_{L_2(\Omega_2)} + \|\Pi_0 w_{2|_\Gamma}\|_{L_2(\Omega_2)} \\
&\leq C\,\mathrm{diam}(\Omega_2)\,\|\nabla w_2\|_{L_2(\Omega_2)} + \big(|\Omega_2|/|\Gamma|\big)^{1/2}\|\Pi_0 w_2\|_{L_2(\Gamma)} \\
&\leq C\,\mathrm{diam}(\Omega_2)\,\big(\|\nabla w_2\|_{L_2(\Omega_2)} + \|\Pi_0 w_2 - \Pi_0 w_1\|_{L_2(\Gamma)} + \|\Pi_0 w_1\|_{L_2(\Gamma)}\big) \\
&\leq C\,\mathrm{diam}(\Omega_2)\,\big(\|\nabla w_2\|_{L_2(\Omega_1)} + \|\Pi_0[w]\|_{L_2(\Gamma)} + \|\nabla w_1\|_{L_2(\Omega_1)}\big),
\end{aligned} \tag{3.8}$$

where in the last step we have used a trace theorem and (3.7). Estimates (3.7) and (3.8) immediately give (3.6). □

Remark 3. *From the proof of Proposition 2, it is clear that γ_0 in the definition of γ has to be chosen larger than $C_I/4$, where C_I is the inverse inequality constant in* (3.2)*, which only depends on the shape regularity of the meshes and on the polynomial approximation degree for the variable u.*

Remark 4. *Notice that the proof of the stability result stated in Proposition 2 only requires that the discretization space Λ^p contains the globally constant functions on Γ, an assumption which is obviously fulfilled by every reasonable choice of Λ^p.*

3.2. **Error Analysis.** We derive error estimates for the method (2.5) in a standard way from the results proven in Proposition 1 and 2.

Theorem 1. *Assume $u_i \in H^{s+1}(\Omega_i)$, $i = 1, 2$, with $s > 1/2$. Denote by T_i^h, $i = 1, 2$, the union of the elements contained in Ω_i and having one side on Γ and, for any $K \in T_1^h \cup T_2^h$, define $\gamma_K := \max_{\boldsymbol{x} \in \partial K \cap \Gamma} \gamma$. We have*

(3.9)

$$|||(u - u^h, \lambda - \lambda^p)||| \leq C\Big(\mathtt{k} \sum_{K \in \mathcal{T}^h} h_K^{2\min\{k,s\}} |u|^2_{H^{s+1}(K)} + \sum_{K \in T_1^h \cup T_2^h} \gamma_K h_K^{2\min\{k,s\}+1} |u|^2_{H^{s+1}(K)} + \sum_j \big(\mathtt{k}^{-1} \ell_j^{2s} p_j^{-2s} + \sup_{\boldsymbol{x} \in \Gamma_j} \gamma^{-1} \ell_j^{2s-1} p_j^{-(2s-1)}\big) |\lambda|^2_{H^{s-1/2}(\Gamma_j)}\Big)^{1/2},$$

with a positive constant C independent of the meshsize and of $\underline{p}$.

We briefly comment on the result of Theorem 1 before showing its proof.

• Assume that $\mathcal{T}^h$ is quasi-uniform and denote by h its characteristic meshsize. Then, from (3.9), it is clear that the optimal choice of the polynomial approximation is $p_j \approx \ell_j / h$, $j = 1, \ldots, n_\Gamma$. In this case, estimate (3.9) becomes

$$|||(u - u^h, \lambda - \lambda^p)||| \leq C \mathtt{k}^{1/2} h^{\min\{k,s\}} \sum_i |u_i|_{H^{s+1}(\Omega_i)}.$$

• Assume that $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ are quasi-uniform and denote by h_1 and h_2 their characteristic meshsizes (here, we are not assuming any bound of h_1 and h_2 in terms of each other). Then, from the definition of the parameter γ, it is clear that, if h_1 and h_2 are very different in size, estimate (3.9) significantly depends on α. If $h_1 \ll h_2$ (resp. $h_2 \ll h_1$), the best result is given for $\alpha = 0$ (resp. $\alpha = 1$). With the optimal choice of the polynomial approximation orders, namely $p_j \approx \ell_j / \max\{h_1, h_2\}$, $j = 1, \ldots, n_\Gamma$, estimate (3.9) becomes

$$|||(u - u^h, \lambda - \lambda^p)||| \leq C \mathtt{k}^{1/2} \sum_i h_i^{\min\{k,s\}} |u_i|_{H^{s+1}(\Omega_i)}.$$

Proof of Theorem 1 Let $\tilde{u}_i^h \in V_i^h$ be the nodal interpolant of u_i in Ω_i, $i = 1, 2$, and let $\tilde{\lambda}^p$ denote the $L_2(\Gamma)$–projection of λ on Λ^p. We decompose, as usual, the error $(u - u^h, \lambda - \lambda^p)$ as $(u - \tilde{u}^h, \lambda - \tilde{\lambda}^p) + (\tilde{u}^h - u^h, \tilde{\lambda}^p - \lambda^p)$. From triangle inequality, Proposition 2, Lemma 1 and Proposition 1, we obtain

(3.10)

$$|||(u - u^h, \lambda - \lambda^p)||| \leq C\left(|||(u - \tilde{u}^h, \lambda - \tilde{\lambda}^p)||| + \|\gamma^{1/2}[u - \tilde{u}^h]\|_{L_2(\Gamma)} + [\![\lambda - \tilde{\lambda}^p]\!]_{-1/2,\Gamma}\right).$$

Then, the error bound (3.9) follows from the approximation estimates of $u - \tilde{u}^h$ and $\lambda - \tilde{\lambda}^p$.

In fact, for the terms involving $u - \tilde{u}^h$, the standard interpolation estimates give

$$\sum_i \|\kappa_i^{1/2}\nabla(u-\tilde{u}^h)\|^2_{L_2(\Omega_i)} + \|a(u-\tilde{u}^h)\|^2_{L_2(\Omega)} \le C\mathtt{k} \sum_{K\in\mathcal{T}^h} h_K^{2\min\{k,s\}} |u|^2_{H^{s+1}(K)}, \tag{3.11}$$

and

$$\begin{aligned} \|\gamma^{1/2}[u-\tilde{u}^h]\|^2_{L_2(\Gamma)} &\le 2\|\gamma^{1/2}(u_1-\tilde{u}_1^h)\|^2_{L_2(\Gamma)} + 2\|\gamma^{1/2}(u_2-\tilde{u}_2^h)\|^2_{L_2(\Gamma)} \\ &\le 2\sum_{K\in T_1^h} \|\gamma_K^{1/2}(u_1-\tilde{u}_1^h)\|^2_{L_2(\partial K)} + 2\sum_{K\in T_2^h} \|\gamma_K^{1/2}(u_2-\tilde{u}_2^h)\|^2_{L_2(\partial K)} \\ &\le C\sum_{K\in T_1^h\cup T_2^h} \gamma_K h_K^{2\min\{k,s\}+1} |u|^2_{H^{s+1}(K)}. \end{aligned} \tag{3.12}$$

For the terms with $\lambda - \tilde{\lambda}^p$, since $\|\lambda-\tilde{\lambda}^p\|_{L_2(\Gamma_j)} = \inf_{\mu^p\in\Lambda^p} \|\lambda-\mu^p\|_{L_2(\Gamma_j)}$, from standard hp–approximation estimates we have

$$\|\lambda-\tilde{\lambda}^p\|_{L_2(\Gamma_j)} \le C\ell_j^t p_j^{-t} |\lambda|_{H^t(\Gamma_j)}, \qquad 0 \le t \le p_j+1, \tag{3.13}$$

which immediately gives

$$\|\gamma^{-1/2}(\lambda-\tilde{\lambda}^p)\|_{L_2(\Gamma)} \le C\Big(\sum_j \sup_{\boldsymbol{x}\in\Gamma_j} (\gamma^{-1}) \ell_j^{2s-1} p_j^{-(2s-1)} |\lambda|^2_{H^{s-1/2}(\Gamma_j)}\Big)^{1/2}. \tag{3.14}$$

It remains to deal with the third term appearing at right-hand side of (3.10). We derive an estimate of $\lambda-\tilde{\lambda}^p$ in the norm $[\!]\cdot[\!]_{-1/2,\Gamma}$ by means of interpolation theory (see, e.g., [8, Theorem 12.2]) between the norms $\|b^{-1}\cdot\|_{L_2(\Gamma)}$ (which is the dual of $\|b\cdot\|_{L_2(\Gamma)}$) and $[\!]\cdot[\!]_{-1,\Gamma}$, where $[\!]\cdot[\!]_{-1,\Gamma}$ denotes the dual norm of $\Big(\|b\cdot\|^2_{L_2(\Gamma)} + \mathrm{diam}(\Omega)^{1/2}|\mathtt{k}^{1/2}\cdot|^2_{H_0^1(\Gamma)}\Big)^{1/2}$. We have

$$\begin{aligned} [\!]\lambda-\tilde{\lambda}^p[\!]_{-1,\Gamma} &\le \sup_{\mu\in H_0^1(\Gamma)} \frac{(\lambda-\tilde{\lambda}^p,\mu)_\Gamma}{\mathrm{diam}(\Omega)^{1/2}|\mathtt{k}^{1/2}\mu|_{H^1(\Gamma)}} \\ &= \mathrm{diam}(\Omega)^{-1/2}\mathtt{k}^{-1/2} \sup_{\mu\in H_0^1(\Gamma)} \frac{(\lambda-\tilde{\lambda}^p,\mu-\tilde{\mu}^p)_\Gamma}{|\mu|_{H^1(\Gamma)}}, \end{aligned} \tag{3.15}$$

where $\tilde{\mu}^p$ denotes the L_2–projection of μ on Λ^p, and $(\cdot,\cdot)_\Gamma$ denotes the usual inner product of $L_2(\Gamma)$. Furthermore,

$$\begin{aligned} (\lambda-\tilde{\lambda}^p,\mu-\tilde{\mu}^p)_\Gamma &= \sum_j (\lambda-\tilde{\lambda}^p,\mu-\tilde{\mu}^p)_{\Gamma_j} \le \sum_j ||\lambda-\tilde{\lambda}^p||_{L_2(\Gamma_j)} ||\mu-\tilde{\mu}^p||_{L_2(\Gamma_j)} \\ &= \sum_j \ell_j p_j^{-1} ||\lambda-\tilde{\lambda}^p||_{L_2(\Gamma_j)} \ell_j^{-1} p_j ||\mu-\tilde{\mu}^p||_{L_2(\Gamma_j)} \\ &\le \Big(\sum_j \ell_j^2 p_j^{-2} ||\lambda-\tilde{\lambda}^p||^2_{L_2(\Gamma_j)}\Big)^{1/2} \Big(\sum_j \ell_j^{-2} p_j^2 ||\mu-\tilde{\mu}^p||^2_{L_2(\Gamma_j)}\Big)^{1/2}. \end{aligned} \tag{3.16}$$

We exploit the estimate (3.13) with $t = s - 1/2$ (resp. $t = 1$) to bound the first (resp. the second) term at right-hand side of (3.16) and obtain

$$
\begin{aligned}
(\lambda - \tilde{\lambda}^p, \mu - \tilde{\mu}^p)_\Gamma &\le C \left(\sum_j \ell_j^{2s+1} p_j^{-(2s+1)} |\lambda|^2_{H^{s-1/2}(\Gamma_j)} \right)^{1/2} \left(\sum_j |\mu|^2_{H^1(\Gamma_j)} \right)^{1/2} \\
&\le C \Big(\sum_j \ell_j^{2s+1} p_j^{-(2s+1)} |\lambda|^2_{H^{s-1/2}(\Gamma_j)} \Big)^{1/2} |\mu|_{H^1(\Gamma)}.
\end{aligned}
\tag{3.17}
$$

From (3.15) and (3.17) we easily get

$$
|||\lambda - \tilde{\lambda}^p|||_{-1,\Gamma} \ \le C \operatorname{diam}(\Omega)^{-1/2} \mathtt{k}^{-1/2} \Big(\sum_j \ell_j^{2s+1} p_j^{-(2s+1)} |\lambda|^2_{H^{s-1/2}(\Gamma_j)} \Big)^{1/2}. \tag{3.18}
$$

Finally, from (3.18) and estimate (3.13) with $t = s - 1/2$, by means of the interpolation theory between function spaces as we have detailed above, we end up with

$$
|||\lambda - \tilde{\lambda}^p|||_{-1/2,\Gamma} \le C \Big(\sum_j \mathtt{k}^{-1} \ell_j^{2s} p_j^{-2s} |\lambda|^2_{H^{s-1/2}(\Gamma_j)} \Big)^{1/2}. \tag{3.19}
$$

Inserting (3.11), (3.12), (3.14) and (3.19) in (3.10) gives the result. □

4. Numerical examples

4.1. **Implementation issues.** Some care has to be taken to avoid ill conditioning of the polynomial approximation on the interface. The mass matrix corresponding to the Taylor polynomial, for example, is the Vandermonde matrix which is notoriously ill conditioned. We have chosen to instead work with Legendre polynomials which are orthogonal in the $L_2([-1,1])$ product. Instead of programming each Legendre polynomial $P_n(x)$ separately for $n = 0, 1, \ldots$, we have used the summation formula

$$
P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!\,(n-k)!\,(n-2\,k)!} x^{n-2\,k},
$$

where $\lfloor \cdot \rfloor$ is the floor function, and integrated (analytically, in advance) products of polynomials and (traces of) test functions, and derivatives thereof, term-wise.

In the examples below, we doubled the polynomial degree for the approximation of λ, for every doubling of the number of nodes on the interface, starting with $p_j = 1$ on the coarsest mesh. The parameter γ was chosen as $\gamma|_{\Gamma_j} = \sqrt{3}/h_{\min}$ where $h_{\min}$ denotes the smallest element size along Γ_j.

4.2. **Interior domain.** We considered the domain $\Omega := (0,3) \times (0,3)$ divided into one interior domain $\Omega_1 := (1,2) \times (1/2, 3/2)$ and one exterior $\Omega_2 := \Omega \setminus \Omega_1$. We set $\kappa_1 = \kappa_2 = 1$,

$$
f = \frac{2\,\pi^2}{9} \sin(\pi x/3) \sin(\pi y/3),
$$

and boundary conditions such that the analytical solution is

$$u = \sin(\pi x/3)\,\sin(\pi y/3).$$

The initial and final meshes, with the elevation of the corresponding approximate solution, are shown in Fig. 2, and in Fig. 3 we show the corresponding convergence behavior, second order convergence in L_2–norm and first order convergence in broken energy norm.

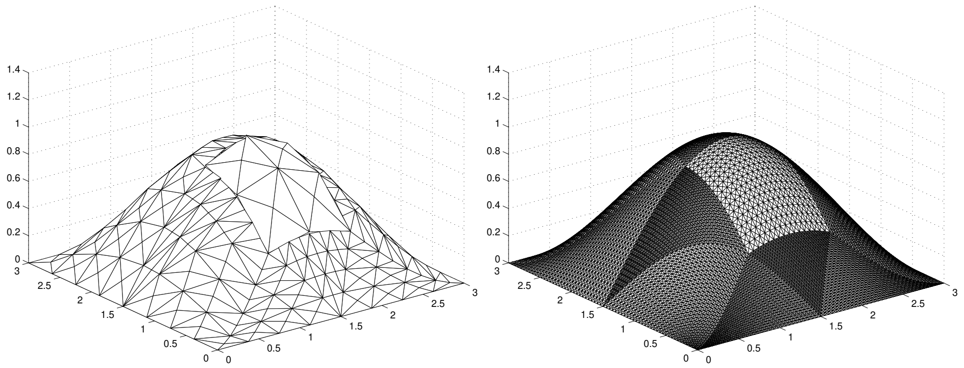

FIGURE 2. First and last mesh in the sequence corresponding to Fig. 3.

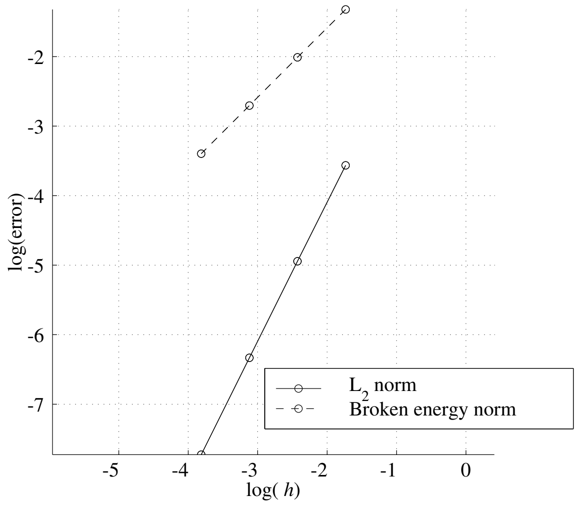

FIGURE 3. Convergence in L_2 and in broken energy norm.

In contrast, the inconsistent method (2.4) gives optimal convergence in broken energy norm but not in L_2–norm, see Fig. 4.

We remark that, in more general cases (e.g., when $\kappa_1 \neq \kappa_2$ and Γ is only Lipschitz continuous), the model problem (2.1) lacks of elliptic regularity. Typically, full elliptic regularity can only be expected if Γ is sufficiently smooth, cf. [6]. Therefore, in general, we cannot expect optimal convergence in L_2–norm even with the consistent method.

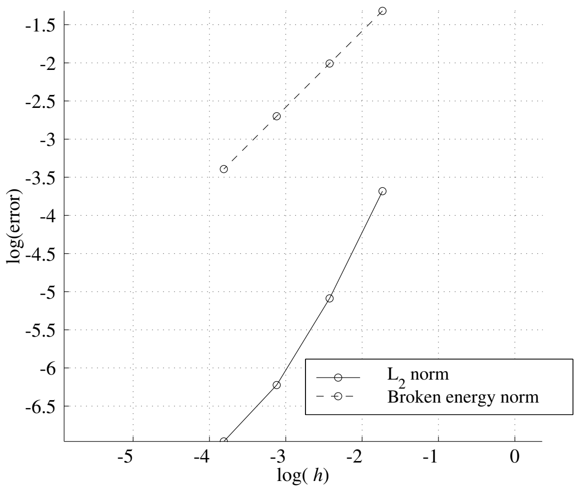

FIGURE 4. Convergence in L_2 and in broken energy norm for the inconsistent method.

4.3. **The case $\kappa_1 \neq \kappa_2$.** Consider solutions to the ordinary differential equation

$$-\sum_i \frac{d}{dx}\left(\kappa_i \frac{du_i}{dx}\right) = 1;\ [u(1/2)] = 0;\ \kappa_1 \frac{du_1}{dx}(1/2) = \kappa_2 \frac{du_2}{dx}(1/2).$$

The domain is $(0,1)$, with an interface at $x = 1/2$. While this is a one-dimensional problem, we solved it numerically in 2D on the domain $(0,1)\times(0,1)$, with zero Neumann boundary conditions at $y = 0$ and $y = 1$. The equation has a closed-form solution that, for homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$, is given by

$$u_1(x) = \frac{(3\,\kappa_1 + \kappa_2)\;x}{4\,\kappa_1^2 + 4\,\kappa_1\,\kappa_2} - \frac{x^2}{2\,\kappa_1}, \quad u_2(x) = \frac{\kappa_2 - \kappa_1 + (3\,\kappa_1 + \kappa_2)\;x}{4\,\kappa_2^2 + 4\,\kappa_1\,\kappa_2} - \frac{x^2}{2\,\kappa_2}.$$

We chose $\kappa_1 = 1/2$, $\kappa_2 = 3$.

The initial and final meshes, with the elevation of the corresponding approximate solution, are shown in Fig. 5, and in Fig. 6 we show the corresponding convergence behavior, again with second order convergence in L_2–norm and first order convergence in broken energy norm.

5. Concluding remarks

We have introduced and analyzed a new stabilized Lagrange multiplier method for interface problems. The basic idea in this method is to avoid integrating products of piecewise function from the two trace meshes, and instead use a global polynomial for the multiplier. It should be pointed out that the stability of our method is not related to the globalness of the polynomial; if a global polynomial is not feasible, some other simple approximation on the interface could be chosen (though we have not analyzed the convergence of other choices). A typical example could be a multiplier space consisting of piecewise constant

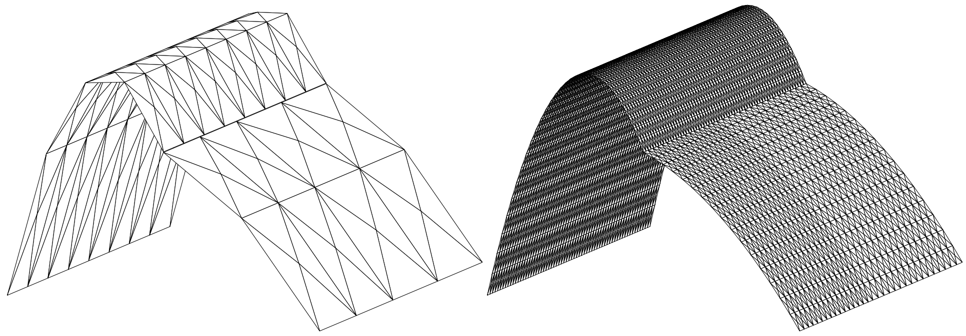

FIGURE 5. First and last mesh in the sequence corresponding to Fig. 6.

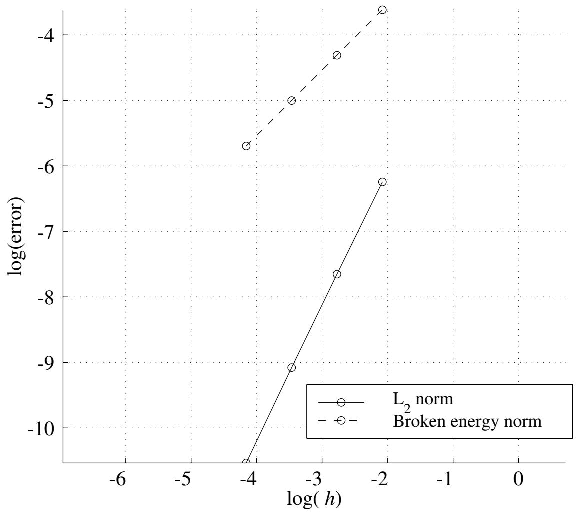

FIGURE 6. Convergence in L_2 and in broken energy norm.

multipliers on a structured Cartesian grid. The main point is avoiding L_2–projections between unrelated and unstructured meshes.

Future work will focus on elastic contact problems where, typically, discrete Lagrange multipliers in the nodes of the trace mesh(es) are used. With a distributed multiplier approach, such as the one suggested here, the need for carefully selecting a master surface (from the point of view of the discretization of the surfaces) will be alleviated since the multiplier space will not be tied directly to the discretization of the contact surfaces.

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