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# A Conservative Flux for the Continuous Galerkin Method based on Discontinuous Enrichment

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**Abstract**

In this paper we develop techniques for computing elementwise conservative approximations of the flux on element boundaries for the continuous Galerkin method. The technique is based on computing a correction of the average normal flux on an edge or face. The correction is a jump in a piecewise constant or linear function. We derive a basic algorithm which is based on solving a global system of equations and a parallel algorithm based on solving local problems on stars. The methods work on meshes with different element types and hanging nodes. We prove existence, uniqueness, and optimal order error estimates. Finally, we illustrate our results by a few numerical examples.

## 1 Introduction

In this paper we develop a technique for computing an elementwise conservative approximation of the normal flux on edges or faces for the continuous Galerkin method. The technique is based on computing a certain correction to the average of the normal fluxes. In the basic case the correction takes the form of a jump in a piecewise constant function and is computed by solving a global linear system of equations with as many degrees of freedom as elements in the mesh. Furthermore, we derive a parallel algorithm where the global system of equations is replaced by localized problems on stars, the set of elements sharing a node. Both methods allow meshes with mixed element type as well as hanging nodes. We prove that both conservative flux approximations are well defined and of optimal order.

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The computed conservative flux is of the same form as in the discontinuous Galerkin (dG) method, see Arnold et al [3], for elliptic problems. We note that the conservation property emanates from the presence of piecewise constant functions in the testspace and thus enriching the continuous test and trial spaces with piecewise constants produces a minimal conservative dG method with optimal order. Our method may be viewed as a postprocessing technique motivated by the minimal conservative dG method.

Techniques for computing discrete conservative fluxes and applications in a posteriori error estimation to manufacture boundary conditions for local elementwise Neumann problems are presented in Kelly [5], Ladeveze [6], Ainsworth and Oden [1], [2], and Verfürth [7]. Conservative fluxes may also be of interest in their own right since we expect them to more correctly represent the true flux.

This paper is organized as follows. In Section 2 we present a model problem, the cG method, and discuss the local conservation property; in Section 3 we present the algorithms for computation of conservative fluxes and prove error estimates; and in Section 4 the analytical results are illustrated by numerical examples.

# 2 The model problem and finite element method

## 2.1 The model problem

We consider the following boundary value problem: find $u : \Omega \to \mathbf{R}$ such that

$$-\nabla \cdot \sigma(u) = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = g_D \quad \text{on } \Gamma_D, \tag{2.2}$$

$$\sigma_n(u) = g_N \quad \text{on } \Gamma_N, \tag{2.3}$$

where $\Omega$ denotes a bounded domain in $\mathbf{R}^d$, $d = 1, 2$, or 3, with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, and the normal flux is defined by $\sigma_n(u) = n \cdot \sigma(u)$, where $n$ is the unit outward normal of $\Gamma$ and the flux

$$\sigma(u) = A\nabla u, \tag{2.4}$$

with $A$ a uniformly positive definite $d \times d$ matrix with bounded entries $a_{ij} \in C(\Omega)$. As is well known (2.1) has a unique solution $u \in H^1$ for each $f \in H^{-1}$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in H^{-1/2}(\Gamma_N)$, for $\Gamma_D \neq \emptyset$; and if $\Gamma_D = \emptyset$, the solution exists and is unique up to a constant, i.e., $u \in H^1/\mathbf{R}$ for $f \in H^{-1}$, $g_N \in H^{-1/2}(\Gamma)$, and the compatibility condition $\int_\Omega f + \int_\Gamma g_N = 0$ is satisfied. We let $\|v\|_{s,\omega}$ and $|v|_{s,\omega}$ denote the standard Sobolev norm and seminorm of order $s$ on the set $\omega$. For brevity we write $\|v\|_{0,\omega} = \|v\|_\omega$.

## 2.2 The mesh

To define the numerical methods we introduce a partition $\mathcal{K} = \{K\}$ of $\Omega$ called the mesh. For simplicity only we assume that the mesh is quasiuniform with meshsize $h$, see [4].

Note that we allow hanging nodes as well as mixed element types, for instance a mesh can contain both triangles and quadrilaterals.

The set of edges in the mesh is denoted by $\mathcal{E} = \{E\}$ and we split $\mathcal{E}$ into three disjoint subsets

$$\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N, \tag{2.5}$$

where $\mathcal{E}_I$ is the set of edges in the interior of $\Omega$, $\mathcal{E}_D$ is the set of edges on the Dirichlet part of the boundary $\Gamma_D$, and $\mathcal{E}_N$ is the set of edges in the Neumann part of the boundary $\Gamma_N$.

To each edge we associate a fixed unit normal $n_E$, such that on the boundary $\Gamma$, $n_E$ is the outward unit normal. We also use the notation $n_K$ for the outward normal of an element $K$.

## 2.3 The standard finite element method

Let $\mathcal{V}_c^p = \mathcal{V}_c^p(\psi)$ denote the space of continuous piecewise polynomials of degree $p$ defined on $\mathcal{K}$, which are equal to $\psi$ on $\Gamma_D$,

$$\mathcal{V}_c^p(\psi) = \{v \in C(\Omega) : v|_{\Gamma_D} = \psi, v|_K \in \mathcal{P}_p(K), K \in \mathcal{K}\}, \tag{2.6}$$

where $\mathcal{P}_p(K)$ is the space of polynomials of degree $p$ defined on $K$. In this note we will be concerned with two cases: $\psi = 0$ for the test space and $\psi = g_D$ for the trial space. We usually write $\mathcal{V}_c^p$ for brevity. The cG method reads: find $U_c \in \mathcal{V}_c^p$ such that

$$a_c(U_c, v) = l_c(v) \quad \text{for all } v \in \mathcal{V}_c^p, \tag{2.7}$$

where

$$a_c(v, w) = (\sigma(v), \nabla w)_\Omega, \tag{2.8}$$

$$l_c(v) = (f, v)_\Omega + (g_N, v)_{\Gamma_N}, \tag{2.9}$$

with $(v, w)_\omega = \int_\omega vw$.

## 2.4 The conservation property

Let $\omega \subset \Omega$ be a subdomain of $\Omega$, and $\chi_\omega$ be the indicator function $\chi_\omega$, defined by $\chi_\omega = 1$ on $\omega$ and 0 on $\Omega \setminus \omega$. Multiplying (2.1) by $\chi_\omega$ and integrating by parts yields the conservation law

$$\int_\omega f + \int_{\partial\omega} \sigma_n(u) = 0. \tag{2.10}$$

Note that $\sigma_n(u) = g_N$ on $\Gamma_N$. This is the fundamental conservation property which we seek to mimic in the discrete case on an element level, i.e., we seek an approximate flux $\Sigma_n(U_c)$ such that $\Sigma_n(U_c) = g_N$ on $\Gamma_N$ and

$$\int_K f + \int_{\partial K} \Sigma_{n_K}(U_c) = 0, \tag{2.11}$$

for all elements $K \in \mathcal{K}$.

# 3 Conservative flux approximations

## 3.1 First derivation

Here we present a derivation of the basic conservative flux. Let $\mathcal{V}_d^0$ denote the space of piecewise constant functions defined on $\mathcal{K}$. We denote the jump at an interior edge $E \in \mathcal{E}_I$ by $[v] = v^+ - v^-$, where $v^{\pm}(x) = \lim_{t\to 0, t>0} v(x \mp n_E t), x \in E$, and $[v] = v^+$ on edges at the boundary $\mathcal{E}_D \cup \mathcal{E}_N$. Then we can state (2.11) in the form

$$\sum_{E\in\mathcal{E}} (\Sigma_{n_E}(U_c), [v])_E + \sum_{K\in\mathcal{K}} (f, v)_K = 0 \quad \text{for all } v \in \mathcal{V}_d^0. \tag{3.1}$$

On each edge $E \in \mathcal{E}_N$, i.e., on the Neumann part of the boundary, we should have

$$\Sigma_{n_E}(U_c) = g_N, \tag{3.2}$$

since here the normal flux is given. Furthermore, on the remaining edges $\mathcal{E}_I \cup \mathcal{E}_D$, $\Sigma_n(U_c)$ should of be an approximation of the exact flux $\sigma_n(u)$ of optimal order. A natural approximation of $\sigma_n(u)$ is the average

$$\langle \sigma_n(U_c) \rangle, \tag{3.3}$$

where $\langle v \rangle = (v^+ + v^-)/2$ on interior edges $E \in \mathcal{E}_I$ and $\langle v \rangle = v^+$ on $E \in \mathcal{E}_D$ (the Dirichlet part of the bondary), which is of optimal order but not elementwise conservative. We thus write

$$\Sigma_n(U_c) = \langle \sigma_n(U_c) \rangle - \Delta_n(U_c), \tag{3.4}$$

where $\Delta_n = \Delta_n(U_c)$ is a correction making the approximate flux conservative. Inserting (3.4) into (3.1) we obtain

$$\begin{aligned} \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (\Delta_{n_E}, [v])_E = &\sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (\langle \sigma_{n_E}(U_c) \rangle, [v])_E \\ &+ \sum_{E\in\mathcal{E}_N} (g_N, v)_E + \sum_{K\in\mathcal{K}} (f, v)_K \quad \text{for all } v \in \mathcal{V}_d^0. \end{aligned} \tag{3.5}$$

This equation suggests that a natural choice of the correction $\Delta_{n_E}$ is

$$\Delta_{n_E} = h^{-1}[V], \tag{3.6}$$

for some $V \in \mathcal{V}_d^0$, to be determined. The scaling with $h$ is motivated by consistency of units. With this choice of $\Delta_{n_E}$ we obtain the problem: find $V \in \mathcal{V}_d^0$ such that

$$\begin{aligned} \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (h^{-1}[V], [v])_E = &\sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} (\langle \sigma_{n_E}(U_c) \rangle, [v])_E \\ &+ \sum_{E\in\mathcal{E}_N} (g_N, v)_E + \sum_{K\in\mathcal{K}} (f, v)_K \quad \text{for all } v \in \mathcal{V}_d^0. \end{aligned} \tag{3.7}$$

This is a linear symmetric system of equations with the number of unknowns equal to the number of elements in the mesh. In Theorem 3.1 we show that (3.7) is solvable and that $[V]$ is uniquely determined on $\mathcal{E}_I \cup \mathcal{E}_D$. Further, setting $v = 1$ on $K$ and 0 on $\Omega \setminus K$, in the righthand side we obtain

$$\sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_{n_E}(U_c) \rangle, [v])_E + \sum_{E \in \mathcal{E}_N} (g_N, v)_E + \sum_{K \in \mathcal{K}} (f, v)_K$$
$$= (\langle \sigma_{n_K}(U_c) \rangle, 1)_{\partial K \setminus \Gamma_N} + (g_N, 1)_{\partial K \cap \Gamma_N} + (f, 1)_K, \quad (3.8)$$

i.e., the residual of the average flux approximation when inserted into the element conservation law. Solving (3.7) a conservative flux may be directly computed using the formula

$$\Sigma_{n_E}(U_c) = \begin{cases} \langle \sigma_{n_E}(U_c) \rangle - h^{-1}[V] & E \in \mathcal{E}_I \cup \mathcal{E}_D, \\ g_N & E \in \mathcal{E}_N. \end{cases} \quad (3.9)$$

## 3.2 The basic algorithm

We summarize our results in the following algorithm:

**Algorithm 1.** Given $U_c \in \mathcal{V}_c^p$ defined by (2.7) a conservative flux $\Sigma_n(U_c)$ can be computed as follows:

- Find $V \in \mathcal{V}_d^0$ such that

$$b_d(V, w) = l_d(w) - a_d(U_c, w) \quad \text{for all } w \in \mathcal{V}_d^0, \quad (3.10)$$

  where

$$b_d(V, w) = \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (h^{-1}[V], [w])_E, \quad (3.11)$$

$$a_d(U_c, w) = - \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_{n_E}(U_c) \rangle, [w])_E, \quad (3.12)$$

$$l_d(w) = \sum_{E \in \mathcal{E}_N} (g_N, w)_E + \sum_{K \in \mathcal{K}} (f, w)_K. \quad (3.13)$$

- Set

$$\Sigma_n(U_c) = \begin{cases} \langle \sigma_n(U_c) \rangle - h^{-1}[V] & E \in \mathcal{E}_I \cup \mathcal{E}_D, \\ g_N & E \in \mathcal{E}_N. \end{cases} \quad (3.14)$$

To formulate an error estimate we define the edge norm

$$\|v\|_{\mathcal{E}}^2 = \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} h \|v\|_E^2. \quad (3.15)$$

**Theorem 3.1** *Problem (3.10) is solvable and* $[V]$ *is uniquely determined. The flux* $\Sigma_n(U_c)$ *defined by Algorithm 1 is elementwise conservative and the error estimate*

$$\|\sigma_n(u) - \Sigma_n(U_c)\|_{\mathcal{E}} \leq Ch^p|u|_{p+1,\Omega}, \tag{3.16}$$

*holds.*

**Proof.** To show that (3.10) has a unique solution we note that if $w \in \mathcal{V}_d^0$ and $b_d(w, w) = 0$ then $w$ is a constant function. If $\Gamma_D$ is nonempty $w$ must be zero and the form is coercive on $\mathcal{V}_d^0$. If $\Gamma_D$ is empty $w$ may be a nonzero constant function but then also the right hand side of (3.10) is zero due to the compatibility condition and thus there is a solution.

Next to prove that $\Sigma_n(U_c)$ is elementwise conservative we note that using the fact that $w \in \mathcal{V}_d^0$ equation (3.10) simplifies to

$$\sum_{\mathcal{E}_I \cup \mathcal{E}_D} (h^{-1}[V], [w])_E = (f, w) + \sum_{\mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_n(U_c) \rangle, [w])_E. \tag{3.17}$$

Rearranging terms the conservation property follows immediately.

Finally, we note that

$$\|\sigma_n(u) - \Sigma_n(U_c)\|_{\mathcal{E}} \leq \|\sigma_n(u) - \langle \sigma_n(U_c) \rangle\|_{\mathcal{E}} + \|h^{-1}[V]\|_{\mathcal{E}}. \tag{3.18}$$

The second term is estimated as follows

$$\begin{align}
\|h^{-1}[V]\|_{\mathcal{E}}^2 &= b_d(V, V) \tag{3.19}\\
&= l_d(V) - a_d(U_c, V) \tag{3.20}\\
&= a_d(u - U_c, V) \tag{3.21}\\
&\leq \|\sigma_n(u) - \langle \sigma_n(U_c) \rangle\|_{\mathcal{E}} \, \|h^{-1}[V]\|_{\mathcal{E}}, \tag{3.22}
\end{align}$$

and thus we conclude that

$$\|h^{-1}[V]\|_{\mathcal{E}} \leq \|\sigma_n(u) - \langle \sigma_n(U_c) \rangle\|_{\mathcal{E}}. \tag{3.23}$$

To complete the proof we note, using a trace inequality elementwise together with the standard a priori error estimate for the finite element method (2.7), that $\|\sigma_n(u) - \langle \sigma_n(U_c) \rangle\|_{\mathcal{E}} \leq Ch^p|u|_{p+1,\Omega}$. $\square$

## 3.3 A parallel algorithm

To derive a parallel version of Algorithm 1 we use the lowest order basis functions on the mesh as a partition of unity to construct local problems. We denote the space of continuous polynomials of lowest order $\mathcal{V}_c^1$ and order all active nodes (all nodes except hanging nodes) from 1 to $N$ and let $\{\varphi_i\}_{i=1}^N$ be the basis functions in $\mathcal{V}_c^1$. Further we let $S_i = \mathrm{supp}(\varphi_i)$

be the star of elements neighboring node $i$. For instance, on triangles the piecewise linear basis functions and on quads the bilinear basis functions.

To construct local problems we wish to replace the test functions in (3.10) by functions of the form $\varphi_i w$ with $w \in \mathcal{V}_d^0$. Note that $\varphi_i w \in \mathcal{V}_d^1$, where $\mathcal{V}_d^1$ is the space of discontinuous piecewise linear functions. For test functions $v \in \mathcal{V}_d^1$ the conservation law corresponding to (3.1) takes the form

$$\sum_{E \in \mathcal{E}} (\Sigma_{n_E}(U_c), [v])_E - \sum_{K \in \mathcal{K}} (A \nabla U_c, \nabla v)_K + \sum_{K \in \mathcal{K}} (f, v)_K = 0 \quad \text{for all } v \in \mathcal{V}_d^1. \tag{3.24}$$

Note the additional second term which vanishes for $v \in \mathcal{V}_d^0$. Motivated by the derivations in Section 3.1 we formulate the following parallel algorithm.

**Algorithm 2:** Given $U_c \in \mathcal{V}_c^p$ defined by (2.7) a conservative flux $\Sigma_n(U_c)$ can be computed as follows:

- For $i = 1, \ldots, N$ determine $V_i \in \mathcal{V}_d^0(S_i) = \{v : v = w|_{S_i}, w \in \mathcal{V}_d^0\}$ such that

$$b_d(V_i, \varphi_i w) = l_d(\varphi_i w) - a_d(U_c, \varphi_i w), \tag{3.25}$$

  for all $w \in \mathcal{V}_d^0(S_i)$. Here

$$b_d(V, v) = \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (h^{-1}[V], [v])_E, \tag{3.26}$$

$$a_d(U_c, v) = \sum_{K \in \mathcal{K}} (A \nabla U_c, \nabla v)_K - \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} (\langle \sigma_{n_E}(U_c) \rangle, [v])_E, \tag{3.27}$$

$$l_d(v) = \sum_{E \in \mathcal{E}_N} (g_N, v)_E + \sum_{K \in \mathcal{K}} (f, v)_K. \tag{3.28}$$

- Set

$$V = \sum_{i=1}^{N} \varphi_i V_i, \tag{3.29}$$

  and

$$\Sigma_n(U_c) = \begin{cases} \langle \sigma_n(U_c) \rangle - h^{-1}[V] & E \in \mathcal{E}_I \cup \mathcal{E}_D, \\ g_N & E \in \mathcal{E}_N. \end{cases} \tag{3.30}$$

**Theorem 3.2** *Problems (3.25) are solvable and $[V]$ is uniquely determined. The flux $\Sigma_n(U_c)$ defined by Algorithm 2 is elementwise conservative and the error estimate*

$$\|\sigma_n(u) - \Sigma_n(U_c)\|_{\mathcal{E}} \leq C h^p |u|_{p+1,\Omega}, \tag{3.31}$$

*holds.*

**Proof.** To prove that the localized problems (3.25) are solvable we first note that if node $i$ resides on $\Gamma_D$ then $b_d(w, \varphi_i w) = 0$ if and only if $w = 0$ and thus the problem is solvable in this case. If node $i$ does not belong to $\Gamma_D$ then $b_d(w, \varphi_i w) = 0$ if and only if $w$ is a constant on $S_i$. In this case we note that the right side of (3.25) satisfies

$$l_d(\varphi_i w) - a_d(U_c, \varphi_i w) = l_c(\varphi_i w) - a_c(U_c, \varphi_i w) = 0, \tag{3.32}$$

and thus problems (3.25) are solvable for all $i = 1, \dots, N$.

Next, to prove that $\Sigma_n(U_c)$ is conservative we note that

$$\begin{aligned} b_d(V, w) &= \sum_{i=1}^{N} b_d(V_i, \varphi_i w) && (3.33)\\ &= \sum_{i=1}^{N} l_d(\varphi_i w) - a_d(U_c, \varphi_i w) && (3.34)\\ &= l_d(w) - a_d(U_c, w), && (3.35) \end{aligned}$$

for all $w \in \mathcal{V}_d^0$. Here we used the fact that $\sum_{i=1}^N \varphi_i = 1$ in the last equality. Now the conservation property follows in the same way as in the proof of Theorem 3.1.

To prove the error estimate we first show the estimate

$$b_d(\varphi_i V_i, V_i) \leq C |||u - U_c|||_{S_i}^2, \quad i = 1, \dots, N, \tag{3.36}$$

where, for convenience, we introduced the norm

$$|||v|||_\omega^2 = \sum_{K \in \mathcal{K}, K \subset \omega} (\sigma(v), \nabla v)_K + \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D, E \subset \omega} h\Big( \|\langle \sigma_n(v) \rangle\|_E^2 + \|h^{-1}[v]\|_E^2 \Big), \tag{3.37}$$

with $\omega \subset \Omega$ a union of elements in $\mathcal{K}$. Without loss of generality we may assume that $V_i$ has average zero on $S_i$. Note that

$$\begin{aligned} b_d(\varphi_i V_i, V_i) &= a_d(u - U_c, \varphi_i V_i) && (3.38)\\ &\leq |||u - U_c|||_{S_i} \, |||\varphi_i V_i|||_{S_i}. && (3.39) \end{aligned}$$

Using finite dimensionality of $\mathcal{V}_d^0(S_i)$ together with scaling we obtain the inequality

$$|||\varphi_i V_i|||_{S_i}^2 \leq C b_d(\varphi_i V_i, V_i), \tag{3.40}$$

and thus (3.36) follows. To estimate the correction we proceed as follows

$$\|h^{-1}[V]\|_{\mathcal{E}}^2 \le C \sum_{i=1}^{N} \|h^{-1}\varphi_i[V_i]\|_{\mathcal{E}}^2 \tag{3.41}$$

$$\le C \sum_{i=1}^{N} b_d(\varphi_i V_i, V_i) \tag{3.42}$$

$$\le C \sum_{i=1}^{N} |||u - U_c|||_{S_i}^2 \tag{3.43}$$

$$\le C|||u - U_c|||_{\Omega}^2, \tag{3.44}$$

where in (3.42) we used the fact that $\varphi_i^2(x) \le \varphi_i(x)$ for all $x \in S_i$ and in (3.43) we employed estimate (3.36). Finally, using a trace inequality elementwise together with the standard a priori error estimate for the finite element method (2.7), we have $|||u - U_c|||_{\Omega} \le Ch^p|u|_{p+1}$.
□

# 4 Numerical examples

We illustrate our estimates on a simple model problem. Consider the Poisson equation (2.1), with $A$ the two by two identity matrix, on the unit square $\Omega = [0, 1]^2$ with homogeneous Dirichlet conditions $g_D = 0$ on the boundary $\Gamma_D = \Gamma$ and right hand side $f$ such that the solution is $u = \sin(\pi x_1)\sin(\pi x_2)$. We solve this problem on a quasiuniform triangulation $\mathcal{K}$ of $\Omega$ using the cG method with polynomials of degree $p = 1, \dots, 4$, and calculate the conservative fluxes using Algorithms 1 and 2.

In Figure 1 we plot the error, measured as in Theorems 3.1 and 3.2, in the average flux (diamond), and the conservative (square – Algorithm 1, star – Algorithm 2) fluxes as functions of the meshsize $h$. We observe that the convergence rates are in agreement with the predictions of Theorems 3.1 and 3.2.

In Figure 2 we plot the exact flux (solid) together with the averaged (solid diamond) and conservative (solid square – Algorithm 1, solid star – Algorithm 2) fluxes on the side from $(0, 0)$ to $(1, 0)$. The mesh is quasiuniform with triangles of approximate size 0.2 and $U_c$ is computed using linears ($p = 1$). Note, in particular, that Algorithm 2 manufactures a piecewise linear conservative flux while the conservative flux computed using Algorithm 1 and the average flux are piecewise constant.

In Figure 3 we select one triangle from the mesh (used in Figure 2) and plot the exact (solid), the averaged (solid diamond), and the conservative (solid star – Algorithm 2) fluxes along the boundary of the triangle.

Finally, we note that all of our numerical examples indicate that the conservative fluxes are certainly in better agreement with the exact flux than the averaged flux.

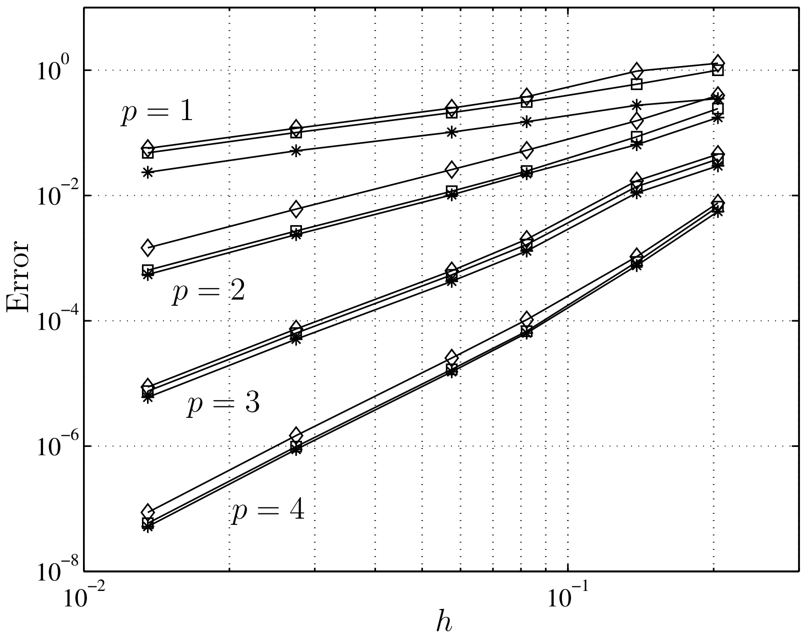


Figure 1: The error in the average (diamond) and conservative (square – Algorithm 1, star – Algorithm 2) fluxes for $p = 1$, 2, 3, and 4.

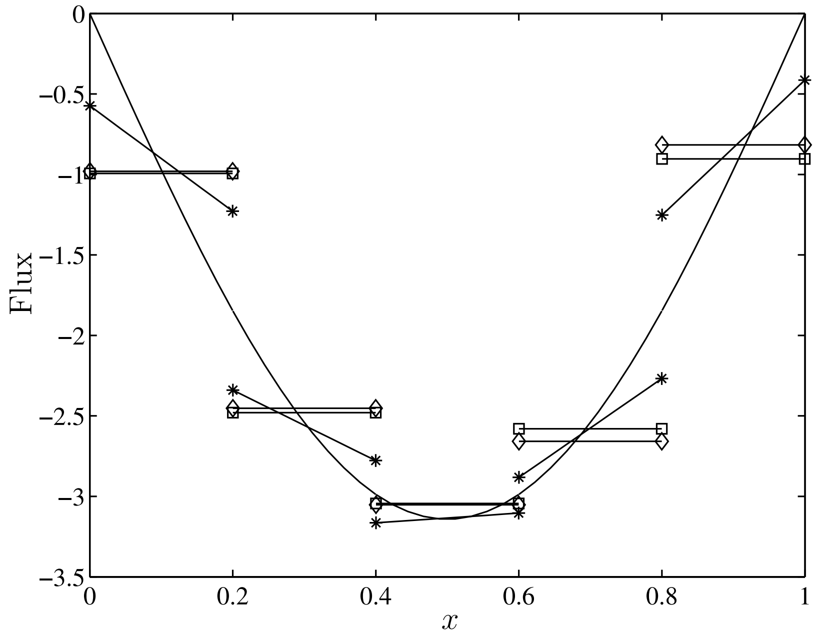


Figure 2: The exact (solid), average (solid diamond) and conservative (solid square – Algorithm 1, solid star – algorithm 2) fluxes for $p = 1$ along one side of the square.

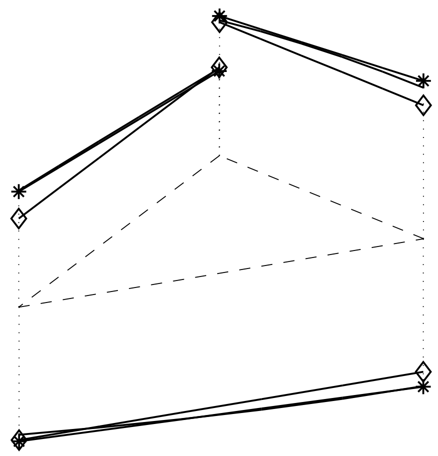

Figure 3: The exact (solid), average (solid diamond) and conservative (solid star – Algorithm 2) fluxes for $p = 1$ along the edges of an element.

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