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PREPRINT 2004–03

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NO 2004–03
ISSN 1404–4382

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Printed in Sweden
Chalmers University of Technology
Göteborg, Sweden 2004

ESTIMATES OF DERIVATIVES AND JUMPS ACROSS ELEMENT BOUNDARIES FOR MULTI-ADAPTIVE GALERKIN SOLUTIONS OF ODES

ANDERS LOGG

Abstract. As an important step in the a priori error analysis of the multi-adaptive Galerkin methods mcG(q) and mdG(q), we prove estimates of derivatives and jumps across element boundaries for the multi-adaptive discrete solutions. The proof is by induction and is based on a new representation formula for the solutions.

1. Introduction

In [3], we proved special interpolation estimates as a preparation for the derivation of a priori error estimates for the multi-adaptive Galerkin methods mcG(q) and mdG(q), presented earlier in [1, 2]. As further preparation, we here derive estimates for derivatives, and jumps in function value and derivatives for the multi-adaptive solutions.

We first derive estimates for the general non-linear problem,

$$\begin{aligned} \dot{u}(t) &= f(u(t), t), \quad t \in (0, T], \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where $u : [0, T] \to \mathbb{R}^N$ is the solution to be computed, $u_0 \in \mathbb{R}^N$ a given initial condition, $T > 0$ a given final time, and $f : \mathbb{R}^N \times (0, T] \to \mathbb{R}^N$ a given function that is Lipschitz-continuous in u and bounded. We also derive estimates for the linear problem,

$$\begin{aligned} \dot{u}(t) + A(t)u(t) &= 0, \quad t \in (0, T], \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

with $A(t)$ a bounded $N \times N$-matrix.

Furthermore, we prove the corresponding estimates for the discrete dual solution Φ, corresponding to (1.1) or (1.2). For the non-linear problem (1.1), the discrete dual solution Φ is defined as a Galerkin solution of the continuous linearized dual problem

$$\begin{aligned} -\dot{\phi}(t) &= J^\top(\pi u, U, t)\phi(t) + g(t), \quad t \in [0, T), \\ \phi(T) &= \psi, \end{aligned} \tag{1.3}$$

Date: February 11, 2004.

Key words and phrases. Multi-adaptivity, individual time steps, local time steps, ODE, continuous Galerkin, discontinuous Galerkin, mcgq, mdgq, a priori error estimates, linear, parabolic.

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with given data $g : [0,T) \to \mathbb{R}^N$ and $\psi \in \mathbb{R}^N$, where

$$J^\top(\pi u, U, t) = \left(\int_0^1 \frac{\partial f}{\partial u}(s\pi u(t) + (1-s)U(t), t)\, ds \right)^\top \tag{1.4}$$

is the transpose of the Jacobian of the right-hand side f, evaluated at an appropriate mean value of the approximate Galerkin solution U of (1.1) and an interpolant πu of the exact solution u. We will use the notation

$$f^*(\phi, \cdot) = J^\top(\pi u, U, \cdot)\phi + g, \tag{1.5}$$

to write the dual problem (1.3) in the form

$$\begin{aligned} -\dot{\phi}(t) &= f^*(\phi(t), t), \quad t \in [0,T), \\ \phi(T) &= \psi. \end{aligned} \tag{1.6}$$

We remind the reader that the discrete dual solution Φ is a Galerkin approximation, given by the mcG$(q)^*$ or mdG$(q)^*$ method defined in [4], of the exact solution ϕ of (1.3), and refer to [4] for the exact definition.

For the linear problem (1.2), the discrete dual solution Φ is defined as a Galerkin solution of the continuous dual problem

$$\begin{aligned} -\dot{\phi}(t) + A^\top(t)\phi(t) &= g, \quad t \in [0,T), \\ \phi(T) &= \psi, \end{aligned} \tag{1.7}$$

or $-\dot{\phi}(t) = f^*(\phi(t), t)$, with the notation $f^*(\phi, \cdot) = -A^\top \phi + g$.

1.1. **Notation.** For a detailed description of the multi-adaptive Galerkin methods, we refer the reader to [1, 2, 6, 4, 5]. In particular, we refer to [1] or [4] for the exact definition of the methods.

The following notation is used throughout this paper: Each component $U_i(t)$, $i = 1, \ldots, N$, of the approximate m(c/d)G(q) solution $U(t)$ of (1.1) is a piecewise polynomial on a partition of $(0,T]$ into M_i subintervals. Subinterval j for component i is denoted by $I_{ij} = (t_{i,j-1}, t_{ij}]$, and the length of the subinterval is given by the local *time step* $k_{ij} = t_{ij} - t_{i,j-1}$. This is illustrated in Figure 1. On each subinterval I_{ij}, $U_i|_{I_{ij}}$ is a polynomial of degree q_{ij} and we refer to $(I_{ij}, U_i|_{I_{ij}})$ as an *element*.

Furthermore, we shall assume that the interval $(0,T]$ is partitioned into blocks between certain synchronized time levels $0 = T_0 < T_1 < \ldots < T_M = T$. We refer to the set of intervals $\mathcal{T}_n$ between two synchronized time levels T_{n-1} and T_n as a *time slab*:

$$\mathcal{T}_n = \{I_{ij} : T_{n-1} \le t_{i,j-1} < t_{ij} \le T_n\}.$$

We denote the length of a time slab by $K_n = T_n - T_{n-1}$. For a given local interval I_{ij}, we denote the time slab $\mathcal{T}$, for which $I_{ij} \in \mathcal{T}$, by $\mathcal{T}(i,j)$.

Since different components use different time steps, a local interval I_{ij} may contain nodal points for other components, that is, some $t_{i'j'} \in (t_{i,j-1}, t_{ij})$. We denote the set of such internal nodes on each local interval I_{ij} by $\mathcal{N}_{ij}$.

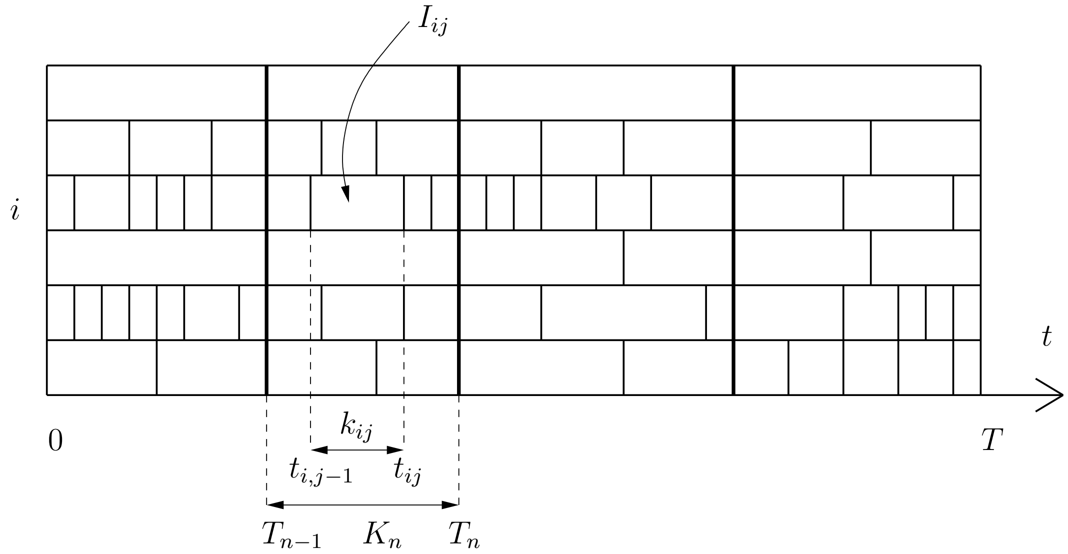

FIGURE 1. Individual partitions of the interval $(0, T]$ for different components. Elements between common synchronized time levels are organized in time slabs. In this example, we have $N = 6$ and $M = 4$.

1.2. **Outline of the paper.** In Section 2, we show that the multi-adaptive Galerkin solutions (including discrete dual solutions) can be expressed as certain interpolants. It is known before [1] that the mcG(q) solution of (1.1) satisfies the relation

$$U_i(t_{ij}) = (u_0)_i + \int_0^{t_{ij}} f_i(U(t), t)\, dt, \quad j = 1, \ldots, M_i, \quad i = 1, \ldots, N, \tag{1.8}$$

with a similar relation for the mdG(q) solution, but this does not hold with t_{ij} replaced by an arbitrary $t \in [0, T]$. However, we prove that

$$U(t) = \pi_{\mathrm{cG}}^{[q]} \left[u_0 + \int_0^{\cdot} f(U(s), s)\, ds \right](t), \tag{1.9}$$

for all $t \in [0, T]$, with $\pi_{\mathrm{cG}}^{[q]}$ a special interpolant. This new way of expressing the multi-adaptive Galerkin solutions is a powerful tool and it is used extensively throughout the remainder of the paper.

In Section 3, we prove a chain rule for higher-order derivatives, which we use in Section 4, together with the representations of Section 2, to prove the desired estimates for the non-linear problem (1.1) by induction. Finally, in Section 5, we prove the corresponding estimates for linear problems.

2. A REPRESENTATION FORMULA FOR THE SOLUTIONS

The proof of estimates for derivatives and jumps of the multi-adaptive Galerkin solutions is based on expressing the solutions as certain interpolants. These representations are obtained as follows. Let U be the mcG(q) or mdG(q) solution of (1.1) and define for $i = 1, \ldots, N$,

$$\tilde{U}_i(t) = u_i(0) + \int_0^t f_i(U(s), s)\, ds. \tag{2.1}$$

Similarly, for Φ the mcG(q)* or mdG(q)* solution of (1.6), we define for $i = 1, \ldots, N$,

$$\tilde{\Phi}_i(t) = \psi_i + \int_t^T f_i^*(\Phi(s), s)\, ds. \tag{2.2}$$

We note that $\dot{\tilde{U}} = f(U, \cdot)$ and $-\dot{\tilde{\Phi}} = f^*(\Phi, \cdot)$.

It now turns out that U can be expressed as an interpolant of $\tilde{U}$. Similarly, Φ can be expressed as an interpolant of $\tilde{\Phi}$. We derive these representations in Theorem 2.1 below for the mcG(q) and mcG(q)* methods, and in Theorem 2.2 for the mdG(q) and mdG(q)* methods. We remind the reader about the special interpolants $\pi_{\mathrm{cG}}^{[q]}$, $\pi_{\mathrm{cG}^*}^{[q]}$, $\pi_{\mathrm{dG}}^{[q]}$, and $\pi_{\mathrm{dG}^*}^{[q]}$, defined in [3].

Theorem 2.1. *The* mcG(q) *solution U of* (1.1) *can expressed in the form*

$$U = \pi_{\mathrm{cG}}^{[q]} \tilde{U}. \tag{2.3}$$

Similarly, the mcG(q)* *solution Φ of* (1.6) *can be expressed in the form*

$$\Phi = \pi_{\mathrm{cG}^*}^{[q]} \tilde{\Phi}, \tag{2.4}$$

that is, $U_i = \pi_{\mathrm{cG}}^{[q_{ij}]} \tilde{U}_i$ and $\Phi_i = \pi_{\mathrm{cG}^*}^{[q_{ij}]} \tilde{\Phi}_i$ on each local interval I_{ij}.

Proof. To prove (2.3), we note that if U is the mcG(q) solution of (1.1), then on each local interval I_{ij}, we have

$$\int_{I_{ij}} \dot{U}_i v_m\, dt = \int_{I_{ij}} f_i(U, \cdot) v_m\, dt, \quad m = 0, \ldots, q_{ij} - 1,$$

with $v_m(t) = ((t - t_{i,j-1})/k_{ij})^m$. On the other hand, by the definition of $\tilde{U}$, we have

$$\int_{I_{ij}} \dot{\tilde{U}}_i v_m\, dt = \int_{I_{ij}} f_i(U, \cdot) v_m\, dt, \quad m = 0, \ldots, q_{ij} - 1.$$

Integrating by parts and subtracting, we obtain

$$-\left[(U_i - \tilde{U}_i) v_m\right]_{t_{i,j-1}}^{t_{ij}} + \int_{I_{ij}} \left(U_i - \tilde{U}_i\right) \dot{v}_m\, dt = 0,$$

and thus, since $U_i(t_{i,j-1}) - \tilde{U}_i(t_{i,j-1}) = U_i(t_{ij}) - \tilde{U}_i(t_{ij}) = 0$,

$$\int_{I_{ij}} \left(U_i - \tilde{U}_i\right) \dot{v}_m\, dt = 0.$$

By the definition of the mcG(q)-interpolant $\pi_{\mathrm{cG}}^{[q]}$, it now follows that $U_i = \pi_{\mathrm{cG}}^{[q_{ij}]} \tilde{U}_i$ on I_{ij}.

To prove (2.4), we note that with Φ the mcG$(q)^*$ solution of (1.6), we have

$$-(\psi, v(T)) + \sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_{I_{ij}} \Phi_i \dot{v}_i \, dt = \int_0^T (f^*(\Phi, \cdot), v) \, dt, \tag{2.5}$$

for all continuous test functions v of order $q = \{q_{ij}\}$ vanishing at $t = 0$. On the other hand, by the definition of $\tilde{\Phi}$, it follows that

$$-\int_{I_{ij}} \dot{\tilde{\Phi}}_i v_i \, dt = \int_{I_{ij}} f_i^*(\Phi, \cdot) v_i \, dt.$$

Integrating by parts, we obtain

$$-\left[\tilde{\Phi}_i v_i\right]_{t_{i,j-1}}^{t_{ij}} + \int_{I_{ij}} \tilde{\Phi}_i \dot{v}_i \, dt = \int_{I_{ij}} f_i^*(\Phi, \cdot) v_i \, dt,$$

and thus

$$-(\psi, v(T)) + \sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_{I_{ij}} \tilde{\Phi}_i \dot{v}_i \, dt = \int_0^T (f^*(\Phi, \cdot), v) \, dt, \tag{2.6}$$

since $v(0) = 0$ and both $\tilde{\Phi}$ and v are continuous. Subtracting (2.5) and (2.6), it now follows that

$$\sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_{I_{ij}} (\Phi_i - \tilde{\Phi}_i) \dot{v}_i \, dt = 0,$$

for all test functions v. We now take $\dot{v}_i = 0$ except on I_{ij}, and $\dot{v}_n = 0$ for $n \neq i$, to obtain

$$\int_{I_{ij}} (\Phi_i - \tilde{\Phi}_i) w \, dt = 0 \quad \forall w \in \mathcal{P}^{q_{ij}-1}(I_{ij}),$$

and so $\Phi_i = P^{[q_{ij}-1]} \tilde{\Phi}_i \equiv \pi_{\mathrm{cG}^*}^{[q_{ij}]} \tilde{\Phi}_i$ on I_{ij}. $\square$

Theorem 2.2. *The* mdG(q) *solution* U *of* (1.1) *can expressed in the form*

$$U = \pi_{\mathrm{dG}}^{[q]} \tilde{U}. \tag{2.7}$$

Similarly, the mdG$(q)^*$ *solution* Φ *of* (1.6) *can be expressed in the form*

$$\Phi = \pi_{\mathrm{dG}^*}^{[q]} \tilde{\Phi}, \tag{2.8}$$

that is, $U_i = \pi_{\mathrm{dG}}^{[q_{ij}]} \tilde{U}_i$ *and* $\Phi_i = \pi_{\mathrm{dG}^*}^{[q_{ij}]} \tilde{\Phi}_i$ *on each local interval* I_{ij}.

Proof. To prove (2.7), we note that if U is the mdG(q) solution of (1.1), then on each local interval I_{ij}, we have

$$\int_{I_{ij}} \dot{U}_i v_m \, dt = \int_{I_{ij}} f_i(U, \cdot) v_m \, dt, \quad m = 1, \ldots, q_{ij},$$

with $v_m(t) = ((t - t_{i,j-1})/k_{ij})^m$. On the other hand, by the definition of $\tilde{U}$, we have

$$\int_{I_{ij}} \dot{\tilde{U}}_i v_m \, dt = \int_{I_{ij}} f_i(U, \cdot) v_m \, dt, \quad m = 1, \ldots, q_{ij}.$$

Integrating by parts and subtracting, we obtain

$$\int_{I_{ij}} \left(U_i - \tilde{U}_i \right) \dot{v}_m \, dt - \left(U_i(t_{ij}^-) - \tilde{U}(t_{ij}) \right) = 0,$$

and thus, since $U_i(t_{ij}^-) = \tilde{U}_i(t_{ij})$,

$$\int_{I_{ij}} \left(U_i - \tilde{U}_i \right) \dot{v}_m \, dt = 0.$$

By the definition of the mdG(q)-interpolant $\pi_{\mathrm{dG}}^{[q]}$, it now follows that $U_i = \pi_{\mathrm{dG}}^{[q_{ij}]} \tilde{U}_i$ on I_{ij}.

The representation (2.8) of the dual solution follows directly, since the mdG(q)* method is identical to the mdG(q) method with time reversed. □

Remark 2.1. *The representations of the multi-adaptive Galerkin solutions as certain interpolants are presented here for the general non-linear problem* (1.1)*, but apply also to the linear problem* (1.2)*.*

3. A chain rule for higher-order derivatives

To estimate higher-order derivatives, we face the problem of taking higher-order derivatives of $f(U(t), t)$ with respect to t. In this section, we derive a generalized version of the chain rule for higher-order derivatives. We also prove a basic estimate for the jump in a composite function.

Lemma 3.1. (Chain rule) *Let $v : \mathbb{R}^N \to \mathbb{R}$ be $p > 0$ times differentiable in all its variables, and let $x : \mathbb{R} \to \mathbb{R}^N$ be p times differentiable, so that*

$$v \circ x : \mathbb{R} \to \mathbb{R} \tag{3.1}$$

is p times differentiable. Furthermore, let $D^n v$ denote the nth order tensor defined by

$$D^n v \, w^1 \cdots w^n = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} \frac{\partial^n v}{\partial x_{i_1} \cdots \partial x_{i_n}} w^1_{i_1} \cdots w^n_{i_n},$$

for $w^1, \ldots, w^n \in \mathbb{R}^N$. Then,

$$\frac{d^p (v \circ x)}{dt^p} = \sum_{n=1}^{p} D^n v(x) \sum_{n_1, \ldots, n_n} C_{p, n_1, \ldots, n_n} x^{(n_1)} \cdots x^{(n_n)}, \tag{3.2}$$

where for each n the sum $\sum_{n_1,\ldots,n_n}$ is taken over $n_1 + \ldots + n_n = p$ with $n_i \geq 1$.

Proof. Repeated use of the chain rule and Leibniz rule gives

$$\begin{aligned}\frac{d^p(v\circ x)}{dt^p} &= \frac{d^{p-1}}{dt^{p-1}}Dv(x)x^{(1)} = \frac{d^{p-2}}{dt^{p-2}}\left[D^2v(x)x^{(1)}x^{(1)} + Dv(x)x^{(2)}\right]\\ &= \frac{d^{p-3}}{dt^{p-3}}\left[D^3v(x)x^{(1)}x^{(1)}x^{(1)} + D^2v(x)x^{(2)}x^{(1)} + \ldots + Dv(x)x^{(3)}\right]\\ &= \sum_{n=1}^{p} D^n v(x) \sum_{n_1,\ldots,n_n} C_{p,n_1,\ldots,n_n} x^{(n_1)}\cdots x^{(n_n)},\end{aligned}$$

where for each n the sum is taken over $n_1+\ldots+n_n = p$ with $n_i \geq 1$. □

To estimate the jump in function value and derivatives for the composite function $v\circ x$, we will need the following lemma.

Lemma 3.2. *With* $[A] = A^+ - A^-$, $\langle A\rangle = (A^+ + A^-)/2$ *and* $|A| = \max(|A^+|, |A^-|)$, *we have*

$$[AB] = [A]\langle B\rangle + \langle A\rangle[B], \tag{3.3}$$

and

$$|[A_1A_2\cdots A_n]| \leq \sum_{i=1}^{n} |[A_i]|\ \Pi_{j\neq i}|A_i|. \tag{3.4}$$

Proof. The proof of (3.3) is straightforward:

$$\begin{aligned}[A]\langle B\rangle + \langle A\rangle[B] &= (A^+ - A^-)(B^+ + B^-)/2 + (A^+ + A^-)(B^+ - B^-)/2\\ &= A^+B^+ - A^-B^- = [AB].\end{aligned}$$

It now follows that

$$\begin{aligned}|[A_1A_2\cdots A_n]| &= |[A_1(A_2\cdots A_n)\ ]| = |\,[A_1]\langle A_2\cdots A_n\ \rangle + \langle A_1\rangle[A_2\cdots A_n]\,|\\ &\leq |[A_1]|\cdot|A_2|\cdots|A_n| + |A_1|\cdot|[A_2\cdots A_n]| \leq \sum_{i=1}^{n}|[A_i]|\ \Pi_{j\neq i}|A_i|.\end{aligned}$$

□

Using Lemma 3.1 and 3.2, we now prove basic estimates of derivatives and jumps for the composite function $v\circ x$. We will use the following notation: For $n \geq 0$, let $\|D^n v\|_{L_\infty(\mathbb{R},l_\infty)}$ be defined by

$$\|D^n v\, w^1\cdots w^n\|_{L_\infty(\mathbb{R})} \leq \|D^n v\|_{L_\infty(\mathbb{R},l_\infty)}\|w^1\|_{l_\infty}\cdots\|w^n\|_{l_\infty} \quad \forall\, w^1,\ldots,w^n \in \mathbb{R}^N, \tag{3.5}$$

with $\|D^n v\|_{L_\infty(\mathbb{R},l_\infty)} = \|v\|_{L_\infty(\mathbb{R})}$ for $n = 0$, and define

$$\|v\|_{D^p(\mathbb{R})} = \max_{n=0,\ldots,p}\|D^n v\|_{L_\infty(\mathbb{R},l_\infty)}. \tag{3.6}$$

Lemma 3.3. *Let $v : \mathbb{R}^N \to \mathbb{R}$ be $p \geq 0$ times differentiable in all its variables, let $x : \mathbb{R} \to \mathbb{R}^N$ be p times differentiable, and let $C_x > 0$ be a constant, such that $\|x^{(n)}\|_{L_\infty(\mathbb{R},l_\infty)} \leq C_x^n$, for $n = 1, \ldots, p$. Then, there is a constant $C = C(p) > 0$, such that*

$$\left\| \frac{d^p(v \circ x)}{dt^p} \right\|_{L_\infty(\mathbb{R})} \leq C\|v\|_{D^p(\mathbb{R})} C_x^p. \tag{3.7}$$

Proof. We first note that for $p = 0$, (3.7) follows directly by the definition of $\|v\|_{D^p(\mathbb{R})}$. For $p > 0$, we obtain by Lemma 3.1,

$$\left| \frac{d^p(v \circ x)}{dt^p} \right| \leq C \sum_{n=1}^{p} \sum_{n_1,\ldots,n_n} \left| D^n\, v(x) x^{(n_1)} \cdots x^{(n_n)} \right| \leq C\|v\|_{D^p(\mathbb{R})} C_x^p.$$

□

Lemma 3.4. *Let $v : \mathbb{R}^N \to \mathbb{R}$ be $p + 1 \geq 1$ times differentiable in all its variables, let $x : \mathbb{R} \to \mathbb{R}^N$ be p times differentiable, except possibly at some $t \in \mathbb{R}$, and let $C_x > 0$ be a constant, such that $\|x^{(n)}\|_{L_\infty(\mathbb{R},l_\infty)} \leq C_x^n$ for $n = 1, \ldots, p$. Then, there is a constant $C = C(p) > 0$, such that*

$$\left| \left[\frac{d^p(v \circ x)}{dt^p} \right]_t \right| \leq C\|v\|_{D^{p+1}(\mathbb{R})} \sum_{n=0}^{p} C_x^{p-n} \|[x^{(n)}]_t\|_{l_\infty}. \tag{3.8}$$

Proof. We first note that for $p = 0$, we have

$$\left| \left[\frac{d^p(v \circ x)}{dt^p} \right]_t \right| = |[(v \circ x)]_t| = \left| v(x(t^+)) - v(x(t^-)) \right| \leq \|Dv\|_{L_\infty(\mathbb{R},l_\infty)} \|[x]_t\|_{l_\infty},$$

and so (3.8) holds for $p = 0$. For $p > 0$, we obtain by Lemma 3.1 and Lemma 3.2,

$$\begin{aligned}
\left| \left[\frac{d^p(v \circ x)}{dt^p} \right]_t \right| &\leq C \sum_{n=1}^{p} \sum_{n_1,\ldots,n_n} \left| \left[D^n v(x)\, x^{(n_1)} \cdots x^{(n_n)} \right]_t \right| \\
&\leq C \sum_{n=1}^{p} \sum_{n_1,\ldots,n_n} \|D^{n+1}v\|_{L_\infty(\mathbb{R},l_\infty)} \|[x]_t\|_{l_\infty} C_x^p + \\
&\quad + \|D^n v\|_{L_\infty(\mathbb{R},l_\infty)} \left(\|[x^{(n_1)}]_t\|_{l_\infty} C_x^{p-n_1} + \ldots + \|[x^{(n_n)}]_t\|_{l_\infty} C_x^{p-n_n} \right) \\
&\leq C\|v\|_{D^{p+1}(\mathbb{R})} \sum_{n=0}^{p} C_x^{p-n} \|[x^{(n)}]_t\|_{l_\infty}.
\end{aligned}$$

□

4. Estimates of derivatives and jumps for the non-linear problem

We now derive estimates of derivatives and jumps for the multi-adaptive solutions of the general non-linear problem (1.1). To obtain the estimates for the multi-adaptive solutions U and Φ, we first derive estimates for the functions $\tilde{U}$ and $\tilde{\Phi}$ defined in Section 2. These estimates are then used to derive estimates for U and Φ.

4.1. **Assumptions.** We make the following basic assumptions: Given a time slab $\mathcal{T}$, assume that for each pair of local intervals I_{ij} and I_{mn} within the time slab, we have

$$q_{ij} = q_{mn} = \bar{q}, \tag{A1}$$

and

$$k_{ij} > \alpha\ k_{mn}, \tag{A2}$$

for some $\bar{q} \geq 0$ and some $\alpha \in (0,1)$. We also assume that the problem (1.1) is autonomous,

$$\frac{\partial f_i}{\partial t} = 0, \quad i = 1, \ldots, N. \tag{A3}$$

Note that dual problem is in general non-autonomous. Furthermore, assume that

$$\|f_i\|_{D^{\bar{q}+1}(\mathcal{T})} < \infty, \quad i = 1, \ldots, N, \tag{A4}$$

and take $\|f\|_{\mathcal{T}} \geq \max_{i=1,\ldots,N} \|f_i\|_{D^{\bar{q}+1}(\mathcal{T})}$, such that

$$\|d^p/dt^p (\partial f/\partial u)^\top (x(t))\|_{l_\infty} \leq \|f\|_{\mathcal{T}} C_x^p, \tag{4.5}$$

for $p = 0, \ldots, \bar{q}$, and

$$\|[d^p/dt^p (\partial f/\partial u)^\top (x(t))]_t\|_{l_\infty} \leq \|f\|_{\mathcal{T}} \sum_{n=0}^{p} C_x^{p-n} \|[x^{(n)}]_t\|_{l_\infty}, \tag{4.6}$$

for $p = 0, \ldots, \bar{q}-1$, with the notation of Lemma 3.3 and Lemma 3.4. Note that assumption (A4) implies that each f_i is bounded by $\|f\|_{\mathcal{T}}$. We further assume that there is a constant $c_k > 0$, such that

$$k_{ij} \|f\|_{\mathcal{T}} \leq c_k, \tag{A5}$$

for each local interval I_{ij}. We summarize the list of assumptions as follows:

(A1) the local orders q_{ij} are equal within each time slab;
(A2) the local time steps k_{ij} are semi-uniform within each time slab;
(A3) f is autonomous;
(A4) f and its derivatives are bounded;
(A5) the local time steps k_{ij} are small.

4.2. **Estimates for U.** To simplify the estimates, we introduce the following notation: For given $p > 0$, let $C_{U,p} \geq \|f\|_{\mathcal{T}}$ be a constant, such that

$$\|U^{(n)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C_{U,p}^n, \quad n = 1, \ldots, p. \tag{4.8}$$

For $p = 0$, we define $C_{U,0} = \|f\|_{\mathcal{T}}$. Temporarily, we will assume that there is a constant $c_k' > 0$, such that for each p,

$$k_{ij} C_{U,p} \leq c_k'. \tag{A5$'$}$$

This assumption will be removed below in Theorem 4.1. In the following lemma, we use assumptions (A1), (A3), and (A4) to derive estimates for $\tilde{U}$ in terms of $C_{U,p}$ and $\|f\|_{\mathcal{T}}$.

Lemma 4.1. (Derivative and jump estimates for $\tilde{U}$) *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.1) *and define $\tilde{U}$ as in* (2.1). *If assumptions* (A1), (A3), *and* (A4) *hold, then there is a constant $C = C(\bar{q}) > 0$, such that*

$$\|\tilde{U}^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_{U,p-1}^{p}, \quad p = 1, \ldots, \bar{q}+1, \tag{4.10}$$

and

$$\|[\tilde{U}^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C \sum_{n=0}^{p-1} C_{U,p-1}^{p-n} \|[U^{(n)}]_{t_{i,j-1}}\|_{l_\infty}, \quad p = 1, \ldots, \bar{q}+1, \tag{4.11}$$

for each local interval I_{ij}, where $t_{i,j-1}$ is an internal node of the time slab $\mathcal{T}$.

Proof. By definition, $\tilde{U}_i^{(p)} = \frac{d^{p-1}}{dt^{p-1}} f_i(U)$, and so the results follow directly by Lemma 3.3 and Lemma 3.4, noting that $\|f\|_{\mathcal{T}} \leq C_{U,p-1}$. □

By Lemma 4.1, we now obtain the following estimate for the size of the jump in function value and derivatives for U.

Lemma 4.2. (Jump estimates for U) *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.1). *If assumptions* (A1)–(A5) *and* (A5′) *hold, then there is a constant $C = C(\bar{q}, c_k, c_k', \alpha) > 0$, such that*

$$\|[U^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C k_{ij}^{r+1-p} C_{U,r}^{r+1}, \quad p = 0, \ldots, r+1, \quad r = 0, \ldots, \bar{q}, \tag{4.12}$$

for each local interval I_{ij}, where $t_{i,j-1}$ is an internal node of the time slab $\mathcal{T}$.

Proof. The proof is by induction. We first note that at $t = t_{i,j-1}$, we have

$$\begin{aligned} [U_i^{(p)}]_t &= \left(U_i^{(p)}(t^+) - \tilde{U}_i^{(p)}(t^+)\right) + \left(\tilde{U}_i^{(p)}(t^+) - \tilde{U}_i^{(p)}(t^-)\right) + \left(\tilde{U}_i^{(p)}(t^-) - U_i^{(p)}(t^-)\right) \\ &\equiv e_+ + e_0 + e_-. \end{aligned}$$

By Theorem 2.1 (or Theorem 2.2), U is an interpolant of $\tilde{U}$ and so, by Theorem 5.2 in [3], we have

$$|e_+| \leq C k_{ij}^{r+1-p} \|\tilde{U}_i^{(r+1)}\|_{L_\infty(I_{ij})} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^{r} k_{ij}^{m-p} |[\tilde{U}_i^{(m)}]_x|,$$

for $p = 0, \ldots, r+1$ and $r = 0, \ldots, \bar{q}$. Note that the second sum starts at $m = 1$ rather than at $m = 0$, since $\tilde{U}$ is continuous. Similarly, we have

$$|e_-| \leq C k_{i,j-1}^{r+1-p} \|\tilde{U}_i^{(r+1)}\|_{L_\infty(I_{i,j-1})} + C \sum_{x \in \mathcal{N}_{i,j-1}} \sum_{m=1}^{r} k_{i,j-1}^{m-p} |[\tilde{U}_i^{(m)}]_x|.$$

To estimate e_0, we note that $e_0 = 0$ for $p = 0$, since $\tilde{U}$ is continuous. For $p = 1, \ldots, \bar{q}+1$, Lemma 4.1 gives

$$|e_0| = |[\tilde{U}_i^{(p)}]_t| \leq C \sum_{n=0}^{p-1} C_{U,p-1}^{p-n} \|[U^{(n)}]_t\|_{l_\infty}.$$

Using assumption (A2), and the estimates for e_+, e_0, and e_-, we obtain for $r = 0$ and $p = 0$,

$$|[U_i]_t| \leq Ck_{ij}\|\dot{\tilde{U}}_i\|_{L_\infty(I_{ij})} + 0 + Ck_{i,j-1}\|\dot{\tilde{U}}_i\|_{L_\infty(I_{i,j-1})} \leq C(1+\alpha^{-1})k_{ij}C_{U,0} = Ck_{ij}C_{U,0}.$$

It now follows by assumption (A5), that for $r = 0$ and $p = 1$,

$$|[\dot{U}_i]_t| \leq C\|\dot{\tilde{U}}_i\|_{L_\infty(I_{ij})} + CC_{U,0}\|[U]_t\|_{l_\infty} + C\|\dot{\tilde{U}}_i\|_{L_\infty(I_{i,j-1})} \leq C(1+k_{ij}C_{U,0})C_{U,0} \leq CC_{U,0}.$$

Thus, (4.12) holds for $r = 0$. Assume now that (4.12) holds for $r = \bar{r} - 1 \geq 0$. Then, by Lemma 4.1 and assumption (A5′), it follows that

$$\begin{aligned}|e_+| &\leq Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1} + C\sum_{x\in\mathcal{N}_{ij}}\sum_{m=1}^{\bar{r}}k_{ij}^{m-p}\sum_{n=0}^{m-1}C_{U,m-1}^{m-n}\|[U^n]_x\|_{l_\infty} \\ &\leq Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1} + C\sum k_{ij}^{m-p}C_{U,m-1}^{m-n}k_{ij}^{(\bar{r}-1)+1-n}C_{U,\bar{r}-1}^{(\bar{r}-1)+1} \\ &\leq Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1}\left(1+\sum(k_{ij}C_{U,\bar{r}-1})^{m-1-n}\right) \leq Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1}.\end{aligned}$$

Similarly, we obtain the estimate $|e_-| \leq Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1}$. Finally, we use Lemma 4.1 and assumption (A5′), to obtain the estimate

$$\begin{aligned}|e_0| &\leq C\sum_{n=0}^{p-1}C_{U,p-1}^{p-n}\|[U^n]_t\|_{l_\infty} \leq C\sum_{n=0}^{p-1}C_{U,p-1}^{p-n}k_{ij}^{(\bar{r}-1)+1-n}C_{U,\bar{r}-1}^{(\bar{r}-1)+1} \\ &= Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1}\sum_{n=0}^{p-1}(k_{ij}C_{U,\bar{r}})^{p-1-n} \leq Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1}.\end{aligned}$$

Summing up, we thus obtain $|[U_i^{(p)}]_t| \leq |e_+| + |e_0| + |e_-| \leq Ck_{ij}^{\bar{r}+1-p}C_{U,\bar{r}}^{\bar{r}+1}$, and so (4.12) follows by induction. □

By Lemma 4.1 and Lemma 4.2, we now obtain the following estimate for derivatives of the solution U.

Theorem 4.1. (Derivative estimates for U) *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.1). *If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|U^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C\|f\|_{\mathcal{T}}^p, \quad p = 1,\dots,\bar{q}. \tag{4.13}$$

Proof. By Theorem 2.1 (or Theorem 2.2), U is an interpolant of $\tilde{U}$ and so, by Theorem 5.2 in [3], we have

$$\|U_i^{(p)}\|_{L_\infty(I_{ij})} = \|(\pi\tilde{U}_i)^{(p)}\|_{L_\infty(I_{ij})} \leq C'\|\tilde{U}_i^{(p)}\|_{L_\infty(I_{ij})} + C'\sum_{x\in\mathcal{N}_{ij}}\sum_{m=1}^{p-1}k_{ij}^{m-p}|[\tilde{U}_i^{(m)}]_x|,$$

for some constant $C' = C'(\bar{q})$. For $p = 1$, we thus obtain the estimate

$$\|\dot{U}_i\|_{L_\infty(I_{ij})} \leq C'\|\dot{\tilde{U}}_i\|_{L_\infty(I_{ij})} = C'\|f_i(U)\|_{L_\infty(I_{ij})} \leq C'\|f\|_{\mathcal{T}},$$

by assumption (A4), and so (4.13) holds for $p = 1$.

For $p = 2, \dots, \bar{q}$, assuming that (A5$'$) holds for $C_{U,p-1}$, we use Lemma 4.1, Lemma 4.2 (with $r = p - 1$), and assumption (A2), to obtain

$$\begin{aligned}
\|U_i^{(p)}\|_{L_\infty(I_{ij})} &\leq CC_{U,p-1}^p + C \sum_{x\in\mathcal{N}_{ij}} \sum_{m=1}^{p-1} k_{ij}^{m-p} \sum_{n=0}^{m-1} C_{U,m-1}^{m-n} \|[U^{(n)}]_x\|_{l_\infty} \\
&\leq CC_{U,p-1}^p + C \sum k_{ij}^{m-p} C_{U,m-1}^{m-n} k_{ij}^{(p-1)+1-n} C_{U,p-1}^{(p-1)+1} \\
&\leq CC_{U,p-1}^p \left(1 + \sum (k_{ij} C_{U,m-1})^{m-n}\right) \leq CC_{U,p-1}^p,
\end{aligned}$$

where $C = C(\bar{q}, c_k, c_k', \alpha)$. This holds for all components i and all local intervals I_{ij} within the time slab $\mathcal{T}$, and so

$$\|U^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq CC_{U,p-1}^p, \quad p = 1, \dots, \bar{q},$$

where by definition $C_{U,p-1}$ is a constant, such that $\|U^{(n)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C_{U,p-1}^n$ for $n = 1, \dots, p-1$. Starting at $p = 1$, we now define $C_{U,1} = C_1\|f\|_{\mathcal{T}}$ with $C_1 = C' = C'(\bar{q})$. It then follows that (A5$'$) holds for $C_{U,1}$ with $c_k' = C'c_k$, and thus

$$\|U^{(2)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq CC_{U,2-1}^2 = CC_{U,1}^2 \equiv C_2\|f\|_{\mathcal{T}}^2,$$

where $C_2 = C_2(\bar{q}, c_k, \alpha)$. We may thus define $C_{U,2} = \max(C_1\|f\|_{\mathcal{T}}, \sqrt{C_2}\|f\|_{\mathcal{T}})$. Continuing, we note that (A5$'$) holds for $C_{U,2}$, and thus

$$\|U^{(3)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq CC_{U,3-1}^3 = CC_{U,2}^3 \equiv C_3\|f\|_{\mathcal{T}}^3,$$

where $C_3 = C_3(\bar{q}, c_k, \alpha)$. In this way, we obtain a sequence of constants $C_1, \dots, C_{\bar{q}}$, depending only on $\bar{q}$, c_k, and α, such that $\|U^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C_p\|f\|_{\mathcal{T}}^p$ for $p = 1, \dots, \bar{q}$, and so (4.13) follows if we take $C = \max_{i=1,\dots,\bar{q}} C_i$. □

Having now removed the additional assumption (A5$'$), we obtain the following version of Lemma 4.2.

Theorem 4.2. (Jump estimates for U) *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.1). *If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|[U^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq Ck_{ij}^{\bar{q}+1-p}\|f\|_{\mathcal{T}}^{\bar{q}+1}, \quad p = 0, \dots, \bar{q}, \tag{4.14}$$

for each local interval I_{ij}, where $t_{i,j-1}$ is an internal node of the time slab $\mathcal{T}$.

4.3. **Estimates for Φ.** To obtain estimates corresponding to those of Theorem 4.1 and Theorem 4.2 for the discrete dual solution Φ, we need to consider the fact that $f^* = f^*(\phi, \cdot) = J^\top\phi$ is linear and non-autonomous. To simplify the estimates, we introduce the following notation: For given $p \geq 0$, let $C_{\Phi,p} \geq \|f\|_{\mathcal{T}}$ be a constant, such that

$$\|\Phi^{(n)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C_{\Phi,p}^n\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad n = 0, \dots, p. \tag{4.15}$$

Temporarily, we will assume that for each p there is a constant $c_k'' > 0$, such that

$$k_{ij}C_{\Phi,p} \leq c_k''. \tag{A5$''$}$$

This assumption will be removed below in Theorem 4.3. Now, to obtain estimates for Φ, we first need to derive estimates of derivatives and jumps for J.

Lemma 4.3. *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.1)*, and let πu be an interpolant, of order $\bar{q}$, of the exact solution u of* (1.1)*. If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\left\| \frac{d^p J^\top(\pi u, U)}{dt^p} \right\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C \|f\|_{\mathcal{T}}^{p+1}, \quad p = 0, \ldots, \bar{q}, \tag{4.17}$$

and

$$\left\| \left[\frac{d^p J^\top(\pi u, U)}{dt^p} \right]_{t_{i,j-1}} \right\|_{l_\infty} \leq C k_{ij}^{\bar{q}+1-p} \|f\|_{\mathcal{T}}^{\bar{q}+2}, \quad p = 0, \ldots, \bar{q}-1, \tag{4.18}$$

for each local interval I_{ij}, where $t_{i,j-1}$ is an internal node of the time slab $\mathcal{T}$.

Proof. Since f is autonomous by assumption (A3), we have

$$J(\pi u(t), U(t)) = \int_0^1 \frac{\partial f}{\partial u}(s \pi u(t) + (1-s) U(t))\, ds = \int_0^1 \frac{\partial f}{\partial u}(x_s(t))\, ds,$$

with $x_s(t) = s\pi u(t) + (1-s)U(t)$. Noting that $\|u^{(n)}(t)\|_{l_\infty} \leq C\|f\|_{\mathcal{T}}^n$ by (1.1), it follows by Theorem 4.1 and an interpolation estimate, that $\|x_s^{(n)}(t)\|_{l_\infty} \leq C\|f\|_{\mathcal{T}}^n$, and so (4.17) follows by assumption (A4).

At $t = t_{i,j-1}$, we obtain, by Theorem 4.2 and an interpolation estimate,

$$\begin{aligned}
|[x_{si}^{(n)}]_t| &\leq s\ |[(\pi u_i)^{(n)}]_t| + (1-s)\ |[U_i^{(n)}]_t| \leq\ |[(\pi u_i)^{(n)}]_t| +\ |[U_i^{(n)}]_t| \\
&\leq |(\pi u_i)^{(n)}(t^+) - u_i^{(n)}(t)| + |u_i^{(n)}(t) - (\pi u_i)^{(n)}(t^-))| + C k_{ij}^{\bar{q}+1-n} \|f\|_{\mathcal{T}}^{\bar{q}+1} \\
&\leq C k_{ij}^{\bar{q}+1-n} \|u_i^{(\bar{q}+1)}\|_{L_\infty(I_{ij})} + C k_{i,j-1}^{\bar{q}+1-n} \|u_i^{(\bar{q}+1)}\|_{L_\infty(I_{i,j-1})} + C k_{ij}^{\bar{q}+1-n} \|f\|_{\mathcal{T}}^{\bar{q}+1} \\
&\leq C k_{ij}^{\bar{q}+1-n} \|f\|_{\mathcal{T}}^{\bar{q}+1},
\end{aligned}$$

where we have also used assumption (A2). With similar estimates for other components which are discontinuous at $t = t_{i,j-1}$, the estimate (4.18) now follows by assumptions (A4) and (A5). □

Using these estimates for $J^\top$, we now derive estimates for $\tilde{\Phi}$, corresponding to the estimates for $\tilde{U}$ in Lemma 4.1.

Lemma 4.4. (Derivative and jump estimates for $\tilde{\Phi}$) *Let Φ be the* mcG(q)* *or* mdG(q)* *solution of* (1.3) *with $g = 0$, and define $\tilde{\Phi}$ as in* (2.2)*. If assumptions* (A1)–(A5) *and* (A5$''$) *hold, then there is a constant $C = C(\bar{q}, c_k, c_k'', \alpha) > 0$, such that*

$$\|\tilde{\Phi}^{(p)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C C_{\Phi, p-1}^p \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \quad p = 1, \ldots, \bar{q}+1, \tag{4.19}$$

and
(4.20)
$$\|[\tilde{\Phi}^{(p)}]_{t_{ij}}\|_{l_\infty} \leq C k_{ij}^{\bar{q}+2-p} \|f\|_{\mathcal{T}}^{\bar{q}+2} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum_{n=0}^{p-1} \|f\|_{\mathcal{T}}^{p-n} \|[\Phi^{(n)}]_{t_{ij}}\|_{l_\infty}, \quad p = 1, \ldots, \bar{q},$$

for each local interval I_{ij}, where t_{ij} is an internal node of the time slab $\mathcal{T}$.

Proof. By definition, $\dot{\tilde{\Phi}} = -f^*(\Phi, \cdot) = -J(\pi u, U)^\top \Phi$. It follows that

$$\tilde{\Phi}^{(p)} = -\frac{d^{p-1}}{dt^{p-1}} J^\top \Phi = -\sum_{n=0}^{p-1} \binom{p-1}{n} \left(\frac{d^{p-1-n}}{dt^{p-1-n}} J^\top \right) \Phi^{(n)},$$

and so, by Lemma 4.3,

$$\|\tilde{\Phi}^{(p)}(t)\|_{l_\infty} \leq C \sum_{n=0}^{p-1} \|f\|_{\mathcal{T}}^{p-n} C_{\Phi,p-1}^{n} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_{\Phi,p-1}^{p} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)},$$

for $0 \leq p-1 \leq \bar{q}$. To estimate the jump at $t = t_{ij}$, we use Lemma 3.2, Lemma 4.3, and assumption (A5$''$), to obtain

$$\begin{aligned}
\|[\tilde{\Phi}^{(p)}]_t\|_{l_\infty} &\leq C \sum_{n=0}^{p-1} \left\| \left[\left(\frac{d^{p-1-n}}{dt^{p-1-n}} J^\top \right) \Phi^{(n)} \right]_t \right\|_{l_\infty} \\
&\leq C \sum_{n=0}^{p-1} \left(k_{ij}^{\bar{q}+1-(p-1-n)} \|f\|_{\mathcal{T}}^{\bar{q}+2} C_{\Phi,p-1}^{n} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + \|f\|_{\mathcal{T}}^{p-n} \|[\Phi^{(n)}]_t\|_{l_\infty} \right) \\
&\leq C k_{ij}^{\bar{q}+2-p} \|f\|_{\mathcal{T}}^{\bar{q}+2} \sum_{n=0}^{p-1} k_{ij}^{n} C_{\Phi,p-1}^{n} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum_{n=0}^{p-1} \|f\|_{\mathcal{T}}^{p-n} \|[\Phi^{(n)}]_t\|_{l_\infty} \\
&\leq C k_{ij}^{\bar{q}+2-p} \|f\|_{\mathcal{T}}^{\bar{q}+2} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum_{n=0}^{p-1} \|f\|_{\mathcal{T}}^{p-n} \|[\Phi^{(n)}]_t\|_{l_\infty},
\end{aligned}$$

for $0 \leq p-1 \leq \bar{q}-1$. ☐

Our next task is to estimate the jump in the discrete dual solution Φ itself, corresponding to Lemma 4.2.

Lemma 4.5. (Jump estimates for Φ) *Let Φ be the* mcG$(q)^*$ *or* mdG$(q)^*$ *solution of* (1.3) *with $g = 0$. If assumptions* (A1)–(A5) *and* (A5$''$) *hold, then there is a constant $C = C(\bar{q}, c_k, c_k'', \alpha) > 0$, such that*

$$\|[\Phi^{(p)}]_{t_{ij}}\|_{l_\infty} \leq C k_{ij}^{r+1-p} C_{\Phi,r}^{r+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, r+1, \tag{4.21}$$

with $r = 0, \ldots, \bar{q}-1$ for the mcG(q) *method and $r = 0, \ldots, \bar{q}$ for the* mdG(q) *method, for each local interval I_{ij}, where t_{ij} is an internal node of the time slab $\mathcal{T}$.*

Proof. The proof is by induction. We first note that at $t = t_{ij}$, we have

$$\begin{aligned}[\Phi_i^{(p)}]_t &= \Big(\Phi_i^{(p)}(t^+) - \tilde{\Phi}_i^{(p)}(t^+)\Big) + \Big(\tilde{\Phi}_i^{(p)}(t^+) - \tilde{\Phi}_i^{(p)}(t^-)\Big) + \Big(\tilde{\Phi}_i^{(p)}(t^-) - \Phi_i^{(p)}(t^-)\Big) \\ &\equiv e_+ + e_0 + e_-.\end{aligned}$$

By Theorem 2.1 (or Theorem 2.2), Φ is an interpolant of $\tilde{\Phi}$; if Φ is the mcG$(q)^*$ solution, then Φ_i is the $\pi_{\mathrm{cG}^*}^{[q_{ij}]}$-interpolant of $\tilde{\Phi}_i$ on I_{ij}, and if Φ is the mdG$(q)^*$ solution, then Φ_i is the $\pi_{\mathrm{dG}^*}^{[q_{ij}]}$-interpolant of $\tilde{\Phi}_i$. It follows that

$$|e_-| \leq C k_{ij}^{r+1-p} \|\tilde{\Phi}_i^{(r+1)}\|_{L_\infty(I_{ij})} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^{r} k_{ij}^{m-p} |[\tilde{\Phi}_i^{(m)}]_x|, \quad p = 0, \ldots, r+1,$$

where $r = 0, \ldots, \bar{q}-1$ for the mcG$(q)^*$ solution and $r = 0, \ldots, \bar{q}$ for the mdG$(q)^*$solution. Similarly, we have

$$|e_+| \leq C k_{i,j+1}^{r+1-p} \|\tilde{\Phi}_i^{(r+1)}\|_{L_\infty(I_{i,j+1})} + C \sum_{x \in \mathcal{N}_{i,j+1}} \sum_{m=1}^{r} k_{i,j+1}^{m-p} |[\tilde{\Phi}_i^{(m)}]_x|, \quad p = 0, \ldots, r+1.$$

To estimate e_0, we note that $e_0 = 0$ for $p = 0$, since $\tilde{\Phi}$ is continuous. For $p = 1, \ldots, \bar{q}$, Lemma 4.4 gives

$$|e_0| = |[\tilde{\Phi}_i^{(p)}]_t| \leq C k_{ij}^{\bar{q}+2-p} \|f\|_{\mathcal{T}}^{\bar{q}+2} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} + C \sum_{n=0}^{p-1} \|f\|_{\mathcal{T}}^{p-n} \||[\Phi^{(n)}]_t|\|_{l_\infty}. \tag{4.22}$$

Using assumption (A2), and the estimates for e_+, e_0, and e_-, we obtain for $r = 0$ and $p = 0$,

$$\begin{aligned}|[\Phi_i]_t| &\leq C k_{i,j+1} \|\dot{\tilde{\Phi}}_i\|_{L_\infty(I_{i,j+1})} + 0 + C k_{ij} \|\dot{\tilde{\Phi}}_i\|_{L_\infty(I_{ij})} \leq C(\alpha^{-1}+1) k_{ij} C_{\Phi,0} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} \\ &= C k_{ij} C_{\Phi,0} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}.\end{aligned}$$

For $r = 0$ and $p = 1$, it follows by (4.22), noting that $k_{ij}^{\bar{q}+2-1} \|f\|_{\mathcal{T}}^{\bar{q}+2} \leq C \|f\|_{\mathcal{T}} = C C_{\Phi,0}$, and assumption (A2), that $|e_0| \leq C C_{\Phi,0} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} + C \|f\|_{\mathcal{T}} \|[\Phi]_t\|_{l_\infty} \leq C C_{\Phi,0} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}$, and so,

$$|[\dot{\Phi}_i]_t| \leq C \|\dot{\tilde{\Phi}}_i\|_{L_\infty(I_{i,j+1})} + C C_{\Phi,0} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} + C \|\dot{\tilde{\Phi}}_i\|_{L_\infty(I_{ij})} \leq C C_{\Phi,0} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}.$$

Thus, (4.21) holds for $r = 0$. Assume now that (4.21) holds for $r = \bar{r} - 1 \geq 0$. Then, by Lemma 4.4 and assumption (A5), it follows that

$$
\begin{aligned}
|e_-| &\leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\quad + C \sum_{x\in\mathcal{N}_{ij}} \sum_{m=1}^{r} k_{ij}^{m-p} \left(k_{ij}^{\bar{q}+2-m} \|f\|_{\mathcal{T}}^{\bar{q}+2} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + \sum_{n=0}^{m-1} \|f\|_{\mathcal{T}}^{m-n} \|[\Phi^{(n)}]_t\|_{l_\infty} \right) \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\quad + C \sum \left(k_{ij}^{\bar{q}+2-p} \|f\|_{\mathcal{T}}^{\bar{q}+2} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + \sum \|f\|_{\mathcal{T}}^{m-n} k_{ij}^{m-p+(\bar{r}-1)+1-n} C_{\Phi,\bar{r}-1}^{(\bar{r}-1)+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \right) \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\quad + C \sum \left(k_{ij}^{\bar{q}+2-p} \|f\|_{\mathcal{T}}^{\bar{q}+2} + \sum k_{ij}^{\bar{r}+1-p+m-1-n} \|f\|_{\mathcal{T}}^{m-1-n} C_{\Phi,\bar{r}-1}^{\bar{r}+1} \right) \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}.
\end{aligned}
$$

Similarly, we obtain the estimate

$$
|e_+| \leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}.
$$

Again using the assumption that (4.21) holds for $r = \bar{r} - 1$, we obtain

$$
\begin{aligned}
|e_0| &\leq C k_{ij}^{\bar{q}+2-p} \|f\|_{\mathcal{T}}^{\bar{q}+2} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum_{n=0}^{p-1} \|f\|_{\mathcal{T}}^{p-n} k_{ij}^{(\bar{r}-1)+1-n} C_{\Phi,\bar{r}-1}^{(\bar{r}-1)+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}-1}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \left(1 + \sum_{n=0}^{p-1} (k_{ij} \|f\|_{\mathcal{T}})^{p-1-n} \right) \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}-1}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}.
\end{aligned}
$$

We thus have $|[\Phi_i^{(p)}]_t| \leq |e_+| + |e_0| + |e_-| \leq C k_{ij}^{\bar{r}+1-p} C_{\Phi,\bar{r}}^{\bar{r}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}$, and so (4.21) follows by induction. $\square$

Next, we prove an estimate for the derivatives of the discrete dual solution Φ, corresponding to Theorem 4.1.

Theorem 4.3. (Derivative estimates for Φ) *Let Φ be the* mcG$(q)^*$ *or* mdG$(q)^*$ *solution of* (1.3) *with $g = 0$. If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$
\|\Phi^{(p)}\|_{L_\infty(\mathcal{T},\infty)} \leq C \|f\|_{\mathcal{T}}^{p} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}. \tag{4.23}
$$

Proof. By Theorem 2.1 (or Theorem 2.2), Φ is an interpolant of $\tilde{\Phi}$, and so, by Theorem 5.2 in [3], we have

$$
\|\Phi_i^{(p)}\|_{L_\infty(I_{ij})} = \|(\pi\tilde{\Phi}_i)^{(p)}\|_{L_\infty(I_{ij})} \leq C' \|\tilde{\Phi}_i^{(p)}\|_{L_\infty(I_{ij})} + C' \sum_{x\in\mathcal{N}_{ij}} \sum_{m=1}^{p-1} k_{ij}^{m-p} |[\tilde{\Phi}_i^{(m)}\ ]_x|,
$$

for some constant $C' = C'(\bar{q}) > 0$. For $p = 1$, we thus obtain the estimate

$$\|\dot{\Phi}_i\|_{L_\infty(I_{ij})} \leq C'\|\dot{\bar{\Phi}}_i\|_{L_\infty(I_{ij})} = C'\|f_i^*(\Phi)\|_{L_\infty(I_{ij})} = C'\|J^\top \Phi\|_{L_\infty(I_{ij})} \leq C'\|f\|_{\mathcal{T}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)},$$

by assumption (A4), and so (4.23) holds for $p = 1$.

For $p = 2, \ldots, \bar{q}$, assuming that (A5$''$) holds for $C_{\Phi,p-1}$, we use Lemma 4.4, Lemma 4.5 (with $r = p-1$) and assumption (A2), to obtain

$$\begin{aligned}
\|\Phi_i^{(p)}\|_{L_\infty(I_{ij})} &\leq CC_{\Phi,p-1}^p\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\quad + C\sum_{x\in\mathcal{N}_{ij}}\sum_{m=1}^{p-1} k_{ij}^{m-p}\left(k_{ij}^{\bar{q}+2-m}\|f\|_{\mathcal{T}}^{\bar{q}+2}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + \sum_{n=0}^{m-1}\|f\|_{\mathcal{T}}^{m-n}\|[\Phi^{(n)}]_x\|_{l_\infty}\right) \\
&\leq CC_{\Phi,p-1}^p\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + C\sum k_{ij}^{m-p}\|f\|_{\mathcal{T}}^{m-n}k_{ij}^{(p-1)+1-n}C_{\Phi,p-1}^{(p-1)+1}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq CC_{\Phi,p-1}^p\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + CC_{\Phi,p-1}^p\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}\sum(k_{ij}\|f\|_{\mathcal{T}})^{m-n} \\
&\leq CC_{\Phi,p-1}^p\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)},
\end{aligned}$$

where we have used the fact that $k_{ij}^{m-p}k_{ij}^{\bar{q}+2-m}\|f\|_{\mathcal{T}}^{\bar{q}+2} = \|f\|_{\mathcal{T}}^p(k_{ij}\|f\|_{\mathcal{T}})^{\bar{q}+2-p} \leq CC_{\Phi,p-1}^p$, and where $C = C(\bar{q}, c_k, c_k'', \alpha)$. Continuing now in the same way as in the proof of Theorem 4.1, we obtain

$$\|\Phi^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C\|f\|_{\mathcal{T}}^p\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 1, \ldots, \bar{q},$$

for $C = C(\bar{q}, c_k, \alpha)$, which (trivially) holds also when $p = 0$. □

Having now removed the additional assumption (A5$''$), we obtain the following version of Lemma 4.5.

Theorem 4.4. (Jump estimates for Φ) *Let Φ be the* mcG$(q)^*$ *or* mdG$(q)^*$ *solution of* (1.3) *with $g = 0$. If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|[\Phi^{(p)}]_{t_{ij}}\|_{l_\infty} \leq Ck_{ij}^{\bar{q}-p}\|f\|_{\mathcal{T}}^{\bar{q}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}-1, \tag{4.24}$$

for the mcG$(q)^*$ *solution, and*

$$\|[\Phi^{(p)}]_{t_{ij}}\|_{l_\infty} \leq Ck_{ij}^{\bar{q}+1-p}\|f\|_{\mathcal{T}}^{\bar{q}+1}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}, \tag{4.25}$$

for the mdG$(q)^*$ *solution. This holds for each local interval I_{ij}, where t_{ij} is an internal node of the time slab $\mathcal{T}$.*

4.4. **A special interpolation estimate.** In the derivation of a priori error estimates, we face the problem of estimating the interpolation error $\pi\varphi_i - \varphi_i$ on a local interval I_{ij}, where φ_i is defined by

$$\varphi_i = (J^\top(\pi u, u)\Phi)_i = \sum_{l=1}^N J_{li}(\pi u, u)\Phi_l, \quad i = 1, \ldots, N. \tag{4.26}$$

We note that φ_i may be discontinuous within I_{ij}, if other components have nodes within I_{ij}, see Figure 2, since then some Φ_l (or some J_{li}) may be discontinuous within I_{ij}. To

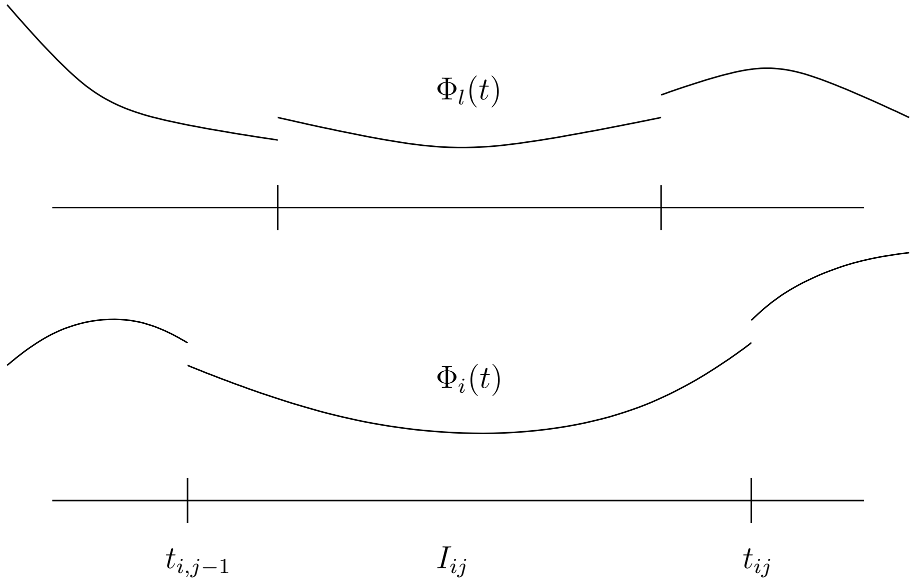

FIGURE 2. If some other component $l \neq i$ has a node within I_{ij}, then Φ_l may be discontinuous within I_{ij}, causing φ_i to be discontinuous within I_{ij}.

estimate the interpolation error, we thus need to estimate derivatives and jumps of φ_i, which requires estimates for both J_{li} and Φ_l.

In Lemma 4.3 we have already proved an estimate for $J^\top$ when f is linearized around πu and U, rather than around πu and u as in (4.26). Replacing U by u, we obtain the following estimate for $J^\top$.

Lemma 4.6. *Let πu be an interpolant, of order $\bar{q}$, of the exact solution u of* (1.1)*. If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\left\| \frac{d^p J^\top(\pi u, u)}{dt^p} \right\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C \|f\|_{\mathcal{T}}^{p+1}, \quad p = 0, \ldots, \bar{q}, \tag{4.27}$$

and

$$\left\| \left[\frac{d^p J^\top(\pi u, u)}{dt^p} \right]_{t_{i,j-1}} \right\|_{l_\infty} \leq C k_{ij}^{\bar{q}+1-p} \|f\|_{\mathcal{T}}^{\bar{q}+2}, \quad p = 0, \ldots, \bar{q}-1, \tag{4.28}$$

for each local interval I_{ij}, where $t_{i,j-1}$ is an internal node of the time slab $\mathcal{T}$.

Proof. See proof of Lemma 4.3. □

From Lemma 4.6 and the estimates for Φ derived in the previous section, we now obtain the following estimates for φ.

Lemma 4.7. (Estimates for φ) *Let φ be defined as in* (4.26)*. If assumptions* (A1)–(A5) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|\varphi_i^{(p)}\|_{L_\infty(I_{ij})} \leq C \|f\|_{\mathcal{T}}^{p+1} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \quad p = 0, \ldots, q_{ij}, \tag{4.29}$$

and

$$|[\varphi_i^{(p)}]_x| \leq C k_{ij}^{r_{ij}-p} \|f\|_{\mathcal{T}}^{r_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \quad \forall x \in \mathcal{N}_{ij}, \quad p = 0, \ldots, q_{ij}-1, \tag{4.30}$$

with $r_{ij} = q_{ij}$ *for the* mcG(q) *method and* $r_{ij} = q_{ij}+1$ *for the* mdG(q) *method. This holds for each local interval* I_{ij} *within the time slab* $\mathcal{T}$.

Proof. Differentiating, we have $\varphi_i^{(p)} = \sum_{n=0}^{p} \binom{p}{n} \frac{d^{p-n} J^\top(\pi u, u)}{dt^{p-n}} \Phi^{(n)}$, and so, by Theorem 4.3 and Lemma 4.6, we obtain

$$\begin{aligned}
\|\varphi_i^{(p)}\|_{L_\infty(I_{ij})} &\leq C \sum_{n=0}^{p} \|f\|_{\mathcal{T}}^{(p-n)+1} \ \|f\|_{\mathcal{T}}^{n} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} = C \sum_{n=0}^{p} \|f\|_{\mathcal{T}}^{p+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&= C \|f\|_{\mathcal{T}}^{p+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}.
\end{aligned}$$

To estimate the jump in $\varphi_i^{(p)}$, we use Lemma 3.2, Theorem 4.3, Theorem 4.4, and Lemma 4.6, to obtain

$$\begin{aligned}
|[\varphi_i^{(p)}]_x| &= \left| \left[\sum_{n=0}^{p} \binom{p}{n} \frac{d^{p-n} J^\top}{dt^{p-n}} \Phi^{(n)} \right]_x \right| \leq C \sum_{n=0}^{p} \left| \left[\frac{d^{p-n} J^\top}{dt^{p-n}} \Phi^{(n)} \right]_x \right| \\
&\leq C \sum_{n=0}^{p} (k_{ij}^{q_{ij}+1-(p-n)} \|f\|_{\mathcal{T}}^{q_{ij}+2} \|f\|_{\mathcal{T}}^{n} + \|f\|_{\mathcal{T}}^{(p-n)+1} k_{ij}^{q_{ij}-n} \|f\|_{\mathcal{T}}^{q_{ij}}) \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq C k_{ij}^{q_{ij}-p} \|f\|_{\mathcal{T}}^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \sum_{n=0}^{p} (k_{ij}\|f\|_{\mathcal{T}})^{n+1} + (k_{ij}\|f\|_{\mathcal{T}})^{p-n} \\
&\leq C k_{ij}^{q_{ij}-p} \|f\|_{\mathcal{T}}^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)},
\end{aligned}$$

for the mcG(q) method. For the mdG(q) method, we obtain one extra power of $k_{ij}\|f\|_{\mathcal{T}}$. ∎

Using the interpolation estimates of [3], together with Lemma 4.7, we now obtain the following important interpolation estimates for φ.

Lemma 4.8. (Interpolation estimates for φ) *Let* φ *be defined as in* (4.26). *If assumptions* (A1)–(A5) *hold, then there is a constant* $C = C(\bar{q}, c_k, \alpha) > 0$, *such that*

$$\|\pi_{\mathrm{cG}}^{[q_{ij}-2]} \varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}-1} \|f\|_{\mathcal{T}}^{q_{ij}} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad q_{ij} = \bar{q} \geq 2, \tag{4.31}$$

and

$$\|\pi_{\mathrm{dG}}^{[q_{ij}-1]} \varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}} \|f\|_{\mathcal{T}}^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad q_{ij} = \bar{q} \geq 1, \tag{4.32}$$

for each local interval I_{ij} *within the time slab* $\mathcal{T}$.

Proof. To prove (4.31), we use Theorem 5.2 in [3], with $r = q_{ij} - 2$ and $p = 0$, together with Lemma 4.7, to obtain

$$\|\pi_{\mathrm{cG}}^{[q_{ij}-2]}\varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq Ck_{ij}^{(q_{ij}-2)+1}\|\varphi_i^{((q_{ij}-2)+1)}\|_{L_\infty(I_{ij})} + C\sum_{x\in\mathcal{N}_{ij}}\sum_{m=0}^{q_{ij}-2} k_{ij}^m\ |[\varphi_i^{(m)}]_x\ |$$

$$\leq Ck_{ij}^{q_{ij}-1}\|f\|_{\mathcal{T}}^{q_{ij}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + C\sum_{x\in\mathcal{N}_{ij}}\sum_{m=0}^{q_{ij}-2} k_{ij}^m k_{ij}^{q_{ij}-m}\|f\|_{\mathcal{T}}^{q_{ij}+1}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}$$

$$= Ck_{ij}^{q_{ij}-1}\|f\|_{\mathcal{T}}^{q_{ij}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} + Ck_{ij}^{q_{ij}}\|f\|_{\mathcal{T}}^{q_{ij}+1}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \leq Ck_{ij}^{q_{ij}-1}\|f\|_{\mathcal{T}}^{q_{ij}}\|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}.$$

The estimate for $\pi_{\mathrm{dG}}^{[q_{ij}-1]}\varphi_i - \varphi_i$ is obtained similarly. □

5. Estimates of derivatives and jumps for linear problems

We now derive estimates for derivatives and jumps for the multi-adaptive solutions of the linear problem (1.2). Assuming that the problem is linear, but non-autonomous, the estimates are obtained in a slightly different way compared to the estimates of the previous section.

5.1. **Assumptions.**

We make the following basic assumptions: Given a time slab $\mathcal{T}$, assume that for each pair of local intervals I_{ij} and I_{mn} within the time slab, we have

$$q_{ij} = q_{mn} = \bar{q}, \tag{B1}$$

and

$$k_{ij} > \alpha\ k_{mn}, \tag{B2}$$

for some $\bar{q} \geq 0$ and some $\alpha \in (0,1)$. Furthermore, assume that A has $\bar{q}-1$ continuous derivatives and let $C_A > 0$ be constant, such that

$$\max\left(\|A^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)}, \|A^{\top(p)}\|_{L_\infty(\mathcal{T},l_\infty)}\right) \leq C_A^{p+1}, \quad p = 0,\ldots,\bar{q}, \tag{B3}$$

for all time slabs $\mathcal{T}$. We further assume that there is a constant $c_k > 0$, such that

$$k_{ij}C_A \leq c_k. \tag{B4}$$

We summarize the list of assumptions as follows:

(B1) the local orders q_{ij} are equal within each time slab;
(B2) the local time steps k_{ij} are semi-uniform within each time slab;
(B3) A and its derivatives are bounded;
(B4) the local time steps k_{ij} are small.

5.2. **Estimates for U and Φ.** To simplify the estimates, we introduce the following notation: For given $p > 0$, let $C_{U,p} \geq C_A$ be a constant, such that

$$\|U^{(n)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C_{U,p}^n \|U\|_{L_\infty(\mathcal{T},l_\infty)}, \quad n = 0, \ldots, p, \tag{5.5}$$

For $p = 0$, we define $C_{U,0} = C_A$. Temporarily, we will assume that there is a constant $c'_k > 0$, such that for each p,

$$k_{ij} C_{U,p} \leq c'_k. \tag{B4$'$}$$

This assumption will be removed below in Theorem 5.1. We similarly define the constant $C_{\Phi,p}$, with $k_{ij} C_{\Phi,p} \leq c'_k$. In the following lemma, we use assumptions (B1) and (B3) to derive estimates for $\tilde{U}$ and $\tilde{\Phi}$.

Lemma 5.1. (Estimates for $\tilde{U}$ and $\tilde{\Phi}$) *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.2) *and define $\tilde{U}$ as in* (2.1). *If assumptions* (B1) *and* (B3) *hold, then there is a constant $C = C(\bar{q}) > 0$, such that*

$$\|\tilde{U}^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_{U,p-1}^p \|U\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 1, \ldots, \bar{q}+1, \tag{5.7}$$

and

$$\|[\tilde{U}^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C \sum_{n=0}^{p-1} C_A^{p-n} \|[U^{(n)}]_{t_{i,j-1}}\|_{l_\infty}, \quad p = 1, \ldots, \bar{q}. \tag{5.8}$$

Similarly, for Φ the mcG(q)* *or* mdG(q)* *solution of* (1.7) *with $g = 0$, and with $\tilde{\Phi}$ defined as in* (2.2), *we obtain*

$$\|\tilde{\Phi}^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_{\Phi,p-1}^p \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 1, \ldots, \bar{q}+1, \tag{5.9}$$

and

$$\|[\tilde{\Phi}^{(p)}]_{t_{ij}}\|_{l_\infty} \leq C \sum_{n=0}^{p-1} C_A^{p-n} \|[\Phi^{(n)}]_{t_{ij}}\|_{l_\infty}, \quad p = 1, \ldots, \bar{q}. \tag{5.10}$$

Proof. By (2.1), it follows that $\dot{\tilde{U}} = -AU$, and so $\tilde{U}^{(p)} = \sum_{n=0}^{p-1} \binom{p-1}{n} A^{(p-1-n)} U^{(n)}$. It now follows by assumptions (B1) and (B3), that

$$\|\tilde{U}^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C \sum_{n=0}^{p-1} C_A^{p-n} \, C_{U,p-1}^n \|U\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_{U,p-1}^p \|U\|_{L_\infty(\mathcal{T},l_\infty)}.$$

Similarly, we obtain $\|[\tilde{U}^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C \sum_{n=0}^{p-1} C_A^{p-n} \|[U^{(n)}]_{t_{i,j-1}}\|_{l_\infty}$. The corresponding estimates for $\tilde{\Phi}$ follow similarly. □

By Lemma 5.1, we now obtain the following estimate for the size of the jump in function value and derivatives for U and Φ.

Lemma 5.2. (Jump estimates for U and Φ) *Let U be the* $\mathrm{mcG}(q)$ *or* $\mathrm{mdG}(q)$ *solution of* (1.2), *and let Φ be the corresponding* $\mathrm{mcG}(q)^*$ *or* $\mathrm{mdG}(q)^*$ *solution of* (1.7) *with $g = 0$. If assumptions* (B1)–(B4) *and* (B4′) *hold, then there is a constant $C = C(\bar{q}, c_k, c_k', \alpha) > 0$, such that*

$$\|[U^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C k_{ij}^{r+1-p} C_{U,r}^{r+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, r+1, \quad r = 0, \ldots, \bar{q}, \tag{5.11}$$

and

$$\|[\Phi^{(p)}]_{t_{ij}}\|_{l_\infty} \leq C k_{ij}^{r+1-p} C_{\Phi,r}^{r+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, r+1, \tag{5.12}$$

with $r = 0, \ldots, \bar{q}-1$ for the $\mathrm{mcG}(q)^*$ *solution and $r = 0, \ldots, \bar{q}$ for the* $\mathrm{mdG}(q)^*$ *solution, for each local interval I_{ij}, where $t_{i,j-1}$ and t_{ij}, respectively, are internal nodes of the time slab $\mathcal{T}$.*

Proof. The proof is by induction and follows those of Lemma 4.2 and Lemma 4.5. We first note that at $t = t_{i,j-1}$, we have

$$\begin{aligned} [U_i^{(p)}]_t &= \left(U_i^{(p)}(t^+) - \tilde{U}_i^{(p)}(t^+)\right) + \left(\tilde{U}_i^{(p)}(t^+) - \tilde{U}_i^{(p)}(t^-)\right) + \left(\tilde{U}_i^{(p)}(t^-) - U_i^{(p)}(t^-)\right) \\ &= e_+ + e_0 + e_-. \end{aligned}$$

Now, U is an interpolant of $\tilde{U}$ and so, by Theorem 5.2 in [3], it follows that

$$|e_+| \leq C k_{ij}^{r+1-p} \|\tilde{U}_i^{(r+1)}\|_{L_\infty(I_{ij})} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^{r} k_{ij}^{m-p} |[\tilde{U}_i^{(m)}]_x|,$$

for $p = 0, \ldots, r+1$ and $r = 0, \ldots, \bar{q}$. Note that the second sum starts at $m = 1$ rather than at $m = 0$, since $\tilde{U}$ is continuous. Similarly, we have

$$|e_-| \leq C k_{i,j-1}^{r+1-p} \|\tilde{U}_i^{(r+1)}\|_{L_\infty(I_{i,j-1})} + C \sum_{x \in \mathcal{N}_{i,j-1}} \sum_{m=1}^{r} k_{i,j-1}^{m-p} |[\tilde{U}_i^{(m)}]_x|.$$

To estimate e_0, we note that $e_0 = 0$ for $p = 0$, since $\tilde{U}$ is continuous. For $p = 1, \ldots, \bar{q}$, Lemma 5.1 gives

$$|e_0| = |[\tilde{U}_i^{(p)}]_t| \leq C \sum_{n=0}^{p-1} C_A^{p-n} \|[U^{(n)}]_t\|_{l_\infty}.$$

Using assumption (B2), and the estimates for e_+, e_0, and e_-, we obtain for $r = 0$ and $p = 0$,

$$\begin{aligned} |[U_i]| &\leq C k_{ij} \|\dot{\tilde{U}}_i\|_{L_\infty(I_{ij})} + 0 + C k_{i,j-1} \|\dot{\tilde{U}}_i\|_{L_\infty(I_{i,j-1})} \leq C(1 + \alpha^{-1}) k_{ij} C_{U,0} \|U\|_{L_\infty(\mathcal{T},l_\infty)} \\ &= C k_{ij} C_{U,0} \|U\|_{L_\infty(\mathcal{T},l_\infty)}. \end{aligned}$$

It now follows by assumption (B4), that for $r = 0$ and $p = 1$,

$$\begin{aligned} |[\dot{U}_i]_t| &\leq C \|\dot{\tilde{U}}_i\|_{L_\infty(I_{ij})} + C C_A \|[U]_t\|_{l_\infty} + C \|\dot{\tilde{U}}_i\|_{L_\infty(I_{i,j-1})} \leq C(1 + k_{ij} C_{U,0}) C_{U,0} \|U\|_{L_\infty(\mathcal{T},l_\infty)} \\ &\leq C C_{U,0} \|U\|_{L_\infty(\mathcal{T},l_\infty)}. \end{aligned}$$

Thus, (5.11) holds for $r = 0$. Assume now that (5.11) holds for $r = \bar{r} - 1 \geq 0$. Then, by Lemma 5.1 and assumption (B4′), it follows that

$$\begin{aligned}
|e_+| &\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum_{x\in\mathcal{N}_{ij}} \sum_{m=1}^{\bar{r}} k_{ij}^{m-p} \sum_{n=0}^{m-1} C_A^{m-n} \|[U^{(n)}]_t\|_{l_\infty} \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum k_{ij}^{m-p} C_A^{m-n} k_{ij}^{(\bar{r}-1)+1-n} C_{U,\bar{r}-1}^{(\bar{r}-1)+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \left(1 + \sum (k_{ij} C_{U,\bar{r}})^{m-1-n}\right) \|U\|_{L_\infty(\mathcal{T},l_\infty)} \leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)}.
\end{aligned}$$

Similarly, we obtain the estimate $|e_-| \leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)}$. Finally, we use Lemma 5.1 and (B4′), to obtain the estimate

$$\begin{aligned}
|e_0| = |[\tilde{U}_i^{(p)}]_t| &\leq C \sum_{n=0}^{p-1} C_A^{p-n} \|[U^{(n)}]_t\|_{l_\infty} \leq C \sum_{n=0}^{p-1} C_A^{p-n} k_{ij}^{(\bar{r}-1)+1-n} C_{U,\bar{r}-1}^{(\bar{r}-1)+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \sum_{n=0}^{p-1} (k_{ij} C_{U,\bar{r}-1})^{p-1-n} \|U\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)}.
\end{aligned}$$

Summing up, we thus obtain $|[U_i^{(p)}]_t| \leq |e_+| + |e_0| + |e_-| \leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)}$, and so (5.11) follows by induction. The estimates for Φ follow similarly. □

Theorem 5.1. (Derivative estimates for U and Φ) *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.2)*, and let Φ be the corresponding* mcG$(q)^*$ *or* mdG$(q)^*$ *solution of* (1.7) *with $g = 0$. If assumptions* (B1)–(B4) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|U^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_A^p \|U\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}, \tag{5.13}$$

and

$$\|\Phi^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_A^p \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}. \tag{5.14}$$

Proof. Since U is an interpolant of $\tilde{U}$, it follows by Theorem 5.2 in [3], that

$$\|U_i^{(p)}\|_{L_\infty(I_{ij})} = \|(\pi\tilde{U}_i)^{(p)}\|_{L_\infty(I_{ij})} \leq C' \|\tilde{U}_i^{(p)}\|_{L_\infty(I_{ij})} + C' \sum_{x\in\mathcal{N}_{ij}} \sum_{m=1}^{p-1} k_{ij}^{m-p} |[\tilde{U}_i^{(m)}]_x|,$$

for some constant $C' = C'(\bar{q})$. For $p = 1$, we thus obtain the estimate

$$\|\dot{U}_i\|_{L_\infty(I_{ij})} \leq C' \|\dot{\tilde{U}}_i\|_{L_\infty(I_{ij})} = C' \|(AU)_i\|_{L_\infty(I_{ij})} \leq C' C_A \|U\|_{L_\infty(\mathcal{T},l_\infty)},$$

and so (5.13) holds for $p = 1$.

For $p = 2, \ldots, \bar{q}$, assuming that (B4′) holds for $C_{U,p-1}$, we use Lemma 5.1, Lemma 5.2 (with $r = p - 1$) and assumption (B2), to obtain

$$\begin{aligned}
\|U_i^{(p)}\|_{L_\infty(I_{ij})} &\leq C C_{U,p-1}^p \|U\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^{p-1} k_{ij}^{m-p} \sum_{n=0}^{m-1} C_A^{m-n} \|[U^{(n)}]_x\|_{l_\infty} \\
&\leq C C_{U,p-1}^p \|U\|_{L_\infty(\mathcal{T},l_\infty)} + C \sum k_{ij}^{m-p} C_A^{m-n} k_{ij}^{(p-1)+1-n} C_{U,p-1}^{(p-1)+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)} \\
&\leq C C_{U,p-1}^p \|U\|_{L_\infty(\mathcal{T},l_\infty)} \left(1 + \sum (k_{ij} C_A)^{m-n}\right) \leq C C_{U,p-1}^p \|U\|_{L_\infty(\mathcal{T},l_\infty)},
\end{aligned}$$

where $C = C(\bar{q}, c_k, c_k', \alpha)$. It now follows in the same way as in the proof of Theorem 4.1, that

$$\|U^{(p)}\|_{L_\infty(\mathcal{T},l_\infty)} \leq C C_A^p \|U\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 1, \ldots, \bar{q},$$

for $C = C(\bar{q}, c_k, \alpha)$, which (trivially) holds also when $p = 0$. The estimate for Φ follows similarly. □

Having now removed the additional assumption (B4′), we obtain the following version of Lemma 5.2.

Theorem 5.2. (Jump estimates for U and Φ) *Let U be the* mcG(q) *or* mdG(q) *solution of* (1.2), *and let Φ be the corresponding* mcG(q)* *or* mdG(q)* *solution of* (1.7) *with $g = 0$. If assumptions* (B1)–(B4) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|[U^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C k_{ij}^{\bar{q}+1-p} C_A^{\bar{q}+1} \|U\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}. \tag{5.15}$$

Furthermore, we have

$$\|[\Phi^{(p)}]_{t_{ij}}\|_{l_\infty} \leq C k_{ij}^{\bar{q}-p} C_A^{\bar{q}} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}, \tag{5.16}$$

for the mcG(q)* *solution and*

$$\|[\Phi^{(p)}]_{t_{ij}}\|_{l_\infty} \leq C k_{ij}^{\bar{q}+1-p} C_A^{\bar{q}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, \bar{q}, \tag{5.17}$$

for the mdG(q)* *solution. This holds for each local interval I_{ij}, where $t_{i,j-1}$ and t_{ij}, respectively, are internal nodes of the time slab $\mathcal{T}$.*

5.3. **A special interpolation estimate.** As for the general non-linear problem, we need to estimate the interpolation error $\pi\varphi_i - \varphi_i$ on a local interval I_{ij}, where φ_i is now defined by

$$\varphi_i = (A^\top \Phi)_i = \sum_{l=1}^{N} A_{li} \Phi_l, \quad i = 1, \ldots, N. \tag{5.18}$$

As noted above, φ_i may be discontinuous within I_{ij}, if I_{ij} contains nodes for other components. We first prove the following estimates for φ.

Lemma 5.3. (Estimates for φ) *Let φ be defined as in* (5.18). *If assumptions* (B1)–(B4) *hold, then there is a constant $C = C(\bar{q}, c_k, \alpha) > 0$, such that*

$$\|\varphi_i^{(p)}\|_{L_\infty(I_{ij})} \leq C C_A^{p+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad p = 0, \ldots, q_{ij}, \tag{5.19}$$

and

$$|[\varphi_i^{(p)}]_x| \leq C k_{ij}^{r_{ij}-p} C_A^{r_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \quad \forall x \in \mathcal{N}_{ij}, \quad p = 0, \ldots, q_{ij}-1, \tag{5.20}$$

with $r_{ij} = q_{ij}$ *for the* mcG(q) *method and* $r_{ij} = q_{ij}+1$ *for the* mdG(q) *method. This holds for each local interval* I_{ij} *within the time slab* $\mathcal{T}$*.*

Proof. Differentiating φ_i, we have $\varphi_i^{(p)} = \frac{d^p}{dt^p}(A^\top \Phi)_i = \sum_{n=0}^p \binom{p}{n} (A^{\top(p-n)} \Phi^{(n)})_i$ and so, by Theorem 5.1, we obtain

$$\|\varphi_i^{(p)}\|_{L_\infty(I_{ij})} \leq C \sum_{n=0}^{p} C_A^{(p-n)+1} C_A^n \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} = C C_A^{p+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}.$$

To estimate the jump in $\varphi_i^{(p)}$, we use Theorem 5.2, to obtain

$$\begin{aligned} |[\varphi_i^{(p)}]_x| &\leq C \sum_{n=0}^{p} |(A^{\top(p-n)} [\Phi^{(n)}]_x)_i| \leq C \sum_{n=0}^{p} C_A^{(p-n)+1} \|[\Phi^{(n)}]_x\|_{l_\infty} \\ &\leq C \sum_{n=0}^{p} C_A^{(p-n)+1} k_{ij}^{\bar{q}-n} C_A^{\bar{q}} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \\ &\leq C k_{ij}^{\bar{q}-p} C_A^{\bar{q}+1} \sum_{n=0}^{p} (k_{ij} C_A)^{p-n} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)} \leq C k_{ij}^{\bar{q}-p} C_A^{\bar{q}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \end{aligned}$$

for the mcG(q) method. For the mdG(q) method, we obtain one extra power of $k_{ij} C_A$. □

Using Lemma 5.3 and the interpolation estimates from [3], we now obtain the following interpolation estimates for φ.

Lemma 5.4. (Interpolation estimates for φ) *Let* φ *be defined as in* (5.18)*. If assumptions* (B1)–(B4) *hold, then there is a constant* $C = C(\bar{q}, c_k, \alpha) > 0$*, such that*

$$\|\pi_{\mathrm{cG}}^{[q_{ij}-2]} \varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}-1} C_A^{q_{ij}} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad q_{ij} = \bar{q} \geq 2, \tag{5.21}$$

and

$$\|\pi_{\mathrm{dG}}^{[q_{ij}-1]} \varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}} C_A^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T},l_\infty)}, \quad q_{ij} = \bar{q} \geq 1, \tag{5.22}$$

for each local interval I_{ij} *within the time slab* $\mathcal{T}$*.*

Proof. See proof of Lemma 4.8. □

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