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A posteriori error analysis of the boundary penalty method

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Abstract

The Boundary Penalty Method enforces Dirichlet boundary conditions weakly by a penalty parameter. We derive a posteriori error estimate of the $L^2(\Omega)$-norm and energy semi-norm for this method and we propose an adaptive strategy to choose the penalty parameter ϵ and the mesh parameter h by equidistributing the error between the terms in the energy semi-norm estimate. Finally, we consider three numerical examples where we successfully use the adaptive algorithm to solve the Poisson equation with both smooth and non-smooth boundary data.

1 Introduction

The Boundary Penalty Method. The Boundary Penalty Method (BPM) has been known and used for more than thirty years. The basic idea is to impose Dirichlet boundary conditions weakly by using Robin type boundary condition with a penalty parameter ϵ. We consider the following model problem: find u such that

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where Ω is a polygonal domain in $\mathbf{R}^d$, $d = 1, 2$ or 3, with boundary Γ. Further $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$ are given data, see [1] for definitions of these spaces. The finite element formulation using BPM [3, 4] now reads: find $U \in V$ such that

$$(\nabla U, \nabla v) + (\epsilon^{-1}U, v)_\Gamma = (f, v) + (\epsilon^{-1}g, v)_\Gamma \quad \text{for all } v \in V, \tag{1.2}$$

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where $(\cdot,\cdot)$ is the $L^2(\Omega)$ scalar product, $(\cdot,\cdot)_\Gamma$ is the $L^2(\Gamma)$ scalar product, and $V \subset H^1(\Omega)$ is the space of continuous piecewise polynomials of degree p with respect to a given triangulation $\mathcal{K} = \{K\}$ of Ω into elements K of diameter h_K. We define the mesh function (mesh parameter) $h(x)$ such that $h(x) = h_K$ when $x \in K$. We assume that the mesh is locally quasi-uniform.

We immediately note that this method is not consistent since u does not solve equation (1.2). Multiplying equation (1.1) with a test function and integrating over the domain using Green's formula gives the following identity for the exact solution u,

$$(\nabla u, \nabla v) - (\partial_n u, v)_\Gamma = (f, v) \quad \text{for all } v \in H^1(\Omega), \tag{1.3}$$

where $\partial_n u = n \cdot \nabla u$ is the normal derivative of u. However, there is a more complicated method for weakly imposing Dirichlet boundary conditions called Nitsche's method [15, 12] which is consistent. The idea in this method is to include the term $(\partial_n U, v)_\Gamma$ that appears in equation (1.3) in equation (1.2) together with a compensating term that makes the method symmetric.

Both BPM and Nitsche's method have been used for problems with interior sub-domain interfaces. One of the first papers on the interior penalty method is Babuška [2] from 1970. In a recent paper [14] this method has been used for gluing together non-matching grids.

There are various reasons for studying the BPM. One is that it allows Dirichlet (ϵ small), Neumann (ϵ large), and Robin (ϵ as a function on Γ) boundary conditions in the same framework. It is also very easy to implement and it has for these reasons been used in many finite element codes over the years. Another reason for studying this method is that it serves as a simpler compliment to Nitsche's method e.g. when solving problems on non-matching grids. As mentioned before Lazarov et.al. [14] chooses this method in their work on non-matching grids.

Previous Work. One of the first works on this subject is Babuška [3] from 1973. His results was then improved and extended among others by Barrett and Elliott [4] during the eighties. Their work are all in an a priori setting and has inspired us to do an a posteriori error analysis of this method. Some important results from these papers are that for piecewise linears $\epsilon = h$ in the boundary penalty formulation, equation (1.2), yields an optimal $H^1(\Omega)$ error estimate but this choice leads to a suboptimal $L^2(\Omega)$ error estimate. As mentioned earlier BPM is not a consistent method i.e. (1.2) will not hold if U is replaced by u. For higher order polynomials this will force the penalty parameter ϵ be proportional to a higher power of h. The reason why $\epsilon \sim h$ is desired is that this choice will not affect the condition number of the stiffness matrix. High condition number leads to slow convergence for iterative solvers. For higher order base functions Nitsche's method is optimal for $\epsilon \sim h$.

As far as we know this is the first a posteriori paper on the boundary penalty method. However, there are several related papers on a posteriori error estimates for discontinuous Galerkin and non-conforming finite element methods [8, 5, 12].

New Contributions. The aim of this paper is to derive an a posteriori error estimate of the error in terms of the mesh parameter h and the penalty parameter ϵ, and based on these results construct an adaptive algorithm to solve problem (1.2) efficiently.

Our main results are the following bounds of the energy and $L^2(\Omega)$ norm of the error $e = u - U$:

$$\|\nabla e\| \leq C\left(\|hR(U)\| + \|g - U\|_{1/2,\Gamma}\right), \tag{1.4}$$

$$\|e\| \leq C\left(\|h^2 R(U)\| + \|g - U\|_{-1/2,\Gamma}\right), \tag{1.5}$$

where $\|\cdot\|_{s,\Gamma}$ is the $H^s(\Gamma)$ norm, $R(U)$ is a computable bound of the residual, $f + \triangle U \in H^{-1}(\Omega)$, on Ω, and C denotes throughout this paper various constants independent of h and ϵ.

To design an adaptive algorithm from the energy semi-norm estimates we need to see explicitly how the a posteriori quantity $\|g - U\|_{1/2,\Gamma}$ depends on ϵ. We introduce P as the $L^2(\Gamma)$ projection onto V and get,

$$\|Pg - U\|_{1/2,\Gamma} \leq C\epsilon \left(\|P(\partial_n U)\|_{1/2,\Gamma} + \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K \right). \tag{1.6}$$

Combining the first part of equation (1.4) and equation (1.6) yields the final error estimate that will be used for the adaptive algorithm,

$$\|\nabla e\| \leq C\left(\|hR(U)\| + \|g - Pg\|_{1/2,\Gamma}\right) + C\epsilon \left(\|P(\partial_n U)\|_{1/2,\Gamma} + \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K \right). \tag{1.7}$$

Obviously there exists an upper bound on ϵ in equation (1.2) for which the approximation gets to poor. We can capture this bound by considering the error estimate in equation (1.7). We also need to impose a lower bound on ϵ for at least two reasons: the condition number of the stiffness matrix grows when ϵ decreases, and we may get undesired oscillations in the solution when solving problems with rough boundary data (see Example 3 in section 4). The conclusion of this discussion is that ϵ needs to be small enough to balance the two terms in equation (1.7) but not smaller.

We also present an estimate of the term $\|Pg - U\|_{-1/2,\Gamma}$ in the $L^2(\Omega)$-norm bound, see equation (1.5), and by using this estimate we get the following bound of the $L^2(\Omega)$-norm of the error,

$$\begin{aligned} \|e\| \leq\ & C\left(\|(h^2 + \epsilon h + \epsilon^2)R(U)\| + \|g - Pg\|_{-1/2,\Gamma} + \epsilon\|g - Pg\|_{1/2,\Gamma}\right) \\ & + \epsilon C\left(\|P(\partial_n U)\|_{-1/2,\Gamma} + \epsilon\|P(\partial_n U)\|_{1/2,\Gamma}\right). \end{aligned} \tag{1.8}$$

Here we see that $\epsilon \sim h$ is not enough to get an optimal order error estimate since the estimate contains the term $\epsilon\|P(\partial_n U)\|_{-1/2,\Gamma}$.

In this work we consider piecewise linear approximations since in this case have an optimal a priori estimate in the energy semi-norm. We are not interested in tracking the constants in the error estimates.

Outline. In Section 2, we present the a posteriori error analysis for control in energy semi-norm and $L^2(\Omega)$-norm. In Section 3 we use the error estimates to derive an adaptive algorithm for choosing the penalty parameter. In Section 4 we present three numerical examples, and finally we present a small summary in Section 5.

2 A Posteriori Error Estimates

2.1 The Error Representation Formula

Subtracting (1.2) from (1.3) yields the error equation

$$(\nabla e, \nabla v) + (\epsilon^{-1} e, v)_\Gamma = (\partial_n u, v)_\Gamma \quad \text{for all } v \in V. \tag{2.1}$$

Green's formula gives,

$$(f + \triangle U, v) + (\partial_n e + \epsilon^{-1} e, v)_\Gamma = (\partial_n u, v)_\Gamma \quad \text{for all } v \in V, \tag{2.2}$$

where the first scalar product is defined in the following way,

$$(f + \triangle U, v) = \sum_K \int_K (f + \triangle U) v \, dx - \sum_K \int_{\partial K \setminus \Gamma} \frac{\partial U}{\partial n_K} v \, ds. \tag{2.3}$$

We also need to take weighted $L^2(\Omega)$ norms of $f + \triangle U$. We define our domain residual according to [10] as a piecewise constant function,

$$R(U) = |f + \triangle U| + \frac{1}{2} \max_{\partial K \setminus \Gamma} h_K^{-1} |[\partial_n U]| \quad \text{on } K \in \mathcal{K}, \tag{2.4}$$

where S are edges on the current element K with boundary ∂K, and $[\cdot]$ is the jump in the function value over the edge. We note that $|(f + \triangle U, v)| \leq \|h^s R(U)\| \|h^{-s} v\|$ for $s \in \mathbf{R}$. Next we introduce a dual problem: find ϕ such that

$$\begin{cases} -\triangle \phi = \psi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma, \end{cases} \tag{2.5}$$

where $\psi \in H^{-1}(\Omega)$. Multiplying (2.5) by the error e and using Green's formula yields,

$$(e, \psi) = (e, -\triangle \phi) = (\nabla e, \nabla \phi) - (e, \partial_n \phi)_\Gamma = (f + \triangle U, \phi) - (g - U, \partial_n \phi)_\Gamma. \tag{2.6}$$

It follows from equation (2.2) that $(f + \triangle U, v) = 0$ for $v \in V$ such that $v = 0$ on Γ. We then get $(f + \triangle U, \pi\phi) = 0$, where $\pi\phi$ is the Scott-Zhang interpolant of ϕ, see [6]. Together this gives,

$$\boxed{(e, \psi) = (f + \triangle U, \phi - \pi\phi) - (g - U, \partial_n \phi)_\Gamma} \tag{2.7}$$

2.2 The Error Estimates

We start this section by proving estimates of the error in energy and $L^2(\Omega)$ norm.

Theorem 2.1 *It holds*

$$\boxed{\|\nabla e\| \leq C\left(\|hR(U)\| + \|g - U\|_{1/2,\Gamma}\right)} \tag{2.8}$$

If we assume that there exists a constant C such that $\|\phi\|_2 \leq C\|\triangle\phi\|$ we also have

$$\boxed{\|e\| \leq C\left(\|h^2R(U)\| + \|g - U\|_{-1/2,\Gamma}\right)} \tag{2.9}$$

Proof. For the energy semi-norm estimate we start from equation (2.7) and let $\psi = -\triangle e$,

$$(e, -\triangle e) = (f + \triangle U, \phi - \pi\phi) - (g - U, \partial_n\phi)_\Gamma. \tag{2.10}$$

We have $(e, -\triangle e) = \|\nabla e\|^2 - (e, \partial_n e)_\Gamma$ and together with equation (2.10) this gives

$$\begin{aligned}
\|\nabla e\|^2 &= (f + \triangle U, \phi - \pi\phi) - (e, \partial_n\phi)_\Gamma + (e, \partial_n e)_\Gamma && (2.11)\\
&\leq C\left(\|hR(U)\|\|\nabla\phi\| + \|e\|_{1/2,\Gamma}\|\partial_n\phi\|_{-1/2,\Gamma} + \|e\|_{1/2,\Gamma}\|\partial_n e\|_{-1/2,\Gamma}\right). && (2.12)
\end{aligned}$$

We recall the trace inequality,

$$\|n \cdot v\|_{-1/2,\Gamma} \leq C \sum_{\partial K\cap\Gamma\neq\emptyset} \left(\|v\|_K + h\|\nabla \cdot v\|_K\right), \tag{2.13}$$

where $\|\cdot\|_K$ is the $L^2(K)$ norm where K refers to elements in the mesh, see ([11], Theorem 2.2) and apply this result twice with $v = \nabla\phi$ and $v = \nabla e$ on equation (2.12) to get,

$$\begin{aligned}
\|\nabla e\|^2 \leq \frac{1}{2}C^2\|hR(U)\|^2 + \frac{1}{2}\|\nabla\phi\|^2 + \|e\|_{1/2,\Gamma} \sum_{\partial K\cap\Gamma\neq\emptyset} \left(\|\nabla e\|_K + \|hR(U)\|_K\right) && (2.14)\\
+ \|e\|_{1/2,\Gamma} \sum_{\partial K\cap\Gamma\neq\emptyset} \left(\|\nabla\phi\|_K + \|hR(U)\|_K\right).
\end{aligned}$$

Next we use the following observation,

$$\|\nabla\phi\|^2 = (-\triangle e, \phi) = (\nabla e, \nabla\phi) \leq \|\nabla e\|\|\nabla\phi\|, \tag{2.15}$$

i.e. $\|\nabla\phi\| \leq \|\nabla e\|$ to get,

$$\begin{aligned}
\|\nabla e\|^2 &\leq C\|hR(U)\|^2 + \frac{1}{2}\|\nabla e\|^2 + 2\|e\|_{1/2,\Gamma}\left(\|\nabla e\| + \|hR(U)\|\right) && (2.16)\\
&\leq C\left(\|hR(U)\|^2 + \|e\|^2_{1/2,\Gamma}\right) + \frac{3}{4}\|\nabla e\|^2. && (2.17)
\end{aligned}$$

Subtracting $3/4\|\nabla e\|^2$ on both sides proves the first part of the theorem.

For the $L^2(\Omega)$ estimate we use $\psi = e/\|e\|$ in (2.7) to get,

$$\|e\| = (e, \psi) = (f + \triangle U, \phi - \pi\phi) - (g - U, \partial_n \phi)_\Gamma. \tag{2.18}$$

Now we use the assumption that there exists a constant C such that $\|\phi\|_2 \leq C\|\triangle\phi\|$ and use the trace inequality $\|\partial_n \phi\|_{1/2} \leq C\|\phi\|_2$ to get

$$\|e\| \leq C\|h^2 R(U)\|\|\phi\|_2 + C\|g - U\|_{-1/2,\Gamma}\|\phi\|_2 \qquad \leq C\left(\|h^2 R(U)\| + \|g - U\|_{-1/2,\Gamma}\right). \tag{2.19}$$

$\square$

In Theorem 2.1 we get bounds with the ϵ dependence hidden. To be able to construct an adaptive algorithm we wish to know how $\|g - U\|_{1/2,\Gamma}$ and $\|g - U\|_{-1/2,\Gamma}$ depends on ϵ. We use the triangle inequality

$$\|g - U\|_{s,\Gamma} \leq \|g - Pg\|_{s,\Gamma} + \|Pg - U\|_{s,\Gamma}, \tag{2.20}$$

for $s = 1/2$ and $s = -1/2$. The first part is independent of ϵ and the second part can be estimated. We start with $\|g - U\|_{1/2,\Gamma}$.

Theorem 2.2 *It holds*

$$\boxed{\|Pg - U\|_{1/2,\Gamma} \leq C\epsilon\left(\|P(\partial_n U)\|_{1/2,\Gamma} + \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K\right)} \tag{2.21}$$

where Pg is the $L^2(\Gamma)$ projection of g onto the restriction of V on the boundary.

Proof. We let $z = P(\epsilon\partial_n U) \in V$ and start by using the triangle inequality,

$$\|Pg - U\|_{1/2,\Gamma} \leq \|z\|_{1/2,\Gamma} + \|Pg - U - z\|_{1/2,\Gamma} \tag{2.22}$$

$$\leq \epsilon\|P(\partial_n U)\|_{1/2,\Gamma} + C\|h^{-1/2}(Pg - U - z)\|_\Gamma, \tag{2.23}$$

where we use an inverse estimate [6] in the second inequality. Next we need to estimate $\|h^{-1/2}(Pg - U - z)\|_\Gamma$.

From the error equation (2.2) we have,

$$-\epsilon(f + \triangle U, v) = (g - U - \epsilon\partial_n U, v)_\Gamma = (Pg - U - z, v)_\Gamma \quad \text{for all } v \in V. \tag{2.24}$$

We let $w \in V$ be equal to zero on interior nodes, $w = P(h^{-1}(Pg - U - z))$ on Γ, and choose $v = w$ in equation (2.24) to get,

$$\|h^{-1/2}(Pg - U - z)\|_\Gamma^2 = (Pg - U - z, w)_\Gamma = (Pg - U - \epsilon\partial_n U, w)_\Gamma = -\epsilon(f + \triangle U, w). \tag{2.25}$$

The right hand side in equation (2.25) can now be estimated in the following way,

$$|(f+\triangle U, w)| \leq C \left(\sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K \right) \|w\| \leq C \left(\sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K \right) \|h^{-1/2}(Pg - U - z)\|_\Gamma. \tag{2.26}$$

We need to take a closer look at the second inequality. Let K be a triangle at the boundary and E the corresponding boundary edge of this triangle. For w as above and the finite element base functions φ_i we have $\|\varphi_i^{1/2} w\|_K^2 \leq C h_K \|\varphi_i^{1/2} w\|_E^2$, by equivalent norms in finite dimensional spaces, and scaling. The assumption of local quasi-uniform mesh gives an estimate of $\|w\|$ in the following way,

$$\|w\|^2 = \int_\Omega \left(\sum_i \varphi_i w \right)^2 \leq C \sum_i \int_\Omega \varphi_i^2 w^2 \leq C \sum_i \sum_{|K \cap \Gamma| \neq 0} \int_K \varphi_i w^2 \tag{2.27}$$

$$\leq C \sum_i \sum_E C h_K \|\varphi_i^{1/2} w\|_E^2 \leq \sum_E C \|h^{1/2} w\|_E^2 = C \|h^{1/2} w\|_\Gamma^2, \tag{2.28}$$

which means that $\|w\| \leq C \|h^{-1/2}(Pg - U - z)\|_\Gamma$. Combining equation (2.25) and equation (2.26) gives

$$\|h^{-1/2}(Pg - U - z)\|_\Gamma \leq C\epsilon \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K. \tag{2.29}$$

Together equation (2.29) and equation (2.22) now gives,

$$\|Pg - U\|_{1/2,\Gamma} \leq \|z\|_{1/2,\Gamma} + C\epsilon \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K, \tag{2.30}$$

which proves the theorem. $\square$

Finally we close this section by finishing the $L^2(\Omega)$-norm estimate in the same way as we did with the energy norm estimate. From Theorem 2.1 we see that we need to estimate $\|g - U\|_{-1/2,\Gamma}$ in terms of the mesh parameter h and ϵ.

Theorem 2.3 *It holds,*

$$\boxed{\|Pg - U\|_{-1/2,\Gamma} \leq \epsilon C \left(\|P(\partial_n U)\|_{-1/2,\Gamma} + \|\nabla e\| + \|hR(U)\| \right)} \tag{2.31}$$

Proof. We start in the same way as in the proof of Theorem 2.2. We let $z = P(\epsilon \partial_n U) \in V$ and use the triangle inequality,

$$\|Pg - U\|_{-1/2,\Gamma} \leq \|z\|_{-1/2,\Gamma} + \|Pg - U - z\|_{-1/2,\Gamma} \tag{2.32}$$

$$\leq \epsilon \|P(\partial_n U)\|_{-1/2,\Gamma} + \|Pg - U - z\|_{-1/2,\Gamma}. \tag{2.33}$$

We study the second term equation (2.33). By definition we have,

$$\|Pg - U - z\|_{-1/2,\Gamma} = \sup_{w\in H^1(\Omega)} \frac{(Pg-U-z,w)_\Gamma}{\|w\|_{1,\Omega}} \tag{2.34}$$

$$= \sup_{w\in H^1(\Omega)} \frac{(Pg-U-z,w-Qw)_\Gamma}{\|w\|_{1,\Omega}} + \sup_{w\in H^1(\Omega)} \frac{(Pg-U-z,Qw)_\Gamma}{\|w\|_{1,\Omega}} \tag{2.35}$$

$$= I + II, \tag{2.36}$$

where Q is the $L^2(\Omega)$-projection onto the finite element space V. We start with the first term I,

$$I \leq \sup_{w\in H^1(\Omega)} \frac{\|h(Pg-U-z)\|_{1/2,\Gamma}\|\frac{1}{h}(w-Qw)\|_{-1/2,\Gamma}}{\|w\|_{1,\Omega}} \tag{2.37}$$

$$\leq \|h(Pg-U-z)\|_{1/2,\Gamma} \sup_{w\in H^1(\Omega)} \frac{\|\frac{1}{h}(w-Qw)\|}{\|w\|_{1,\Omega}} \tag{2.38}$$

$$\leq C\|h(Pg-U-z)\|_{1/2,\Gamma} \tag{2.39}$$

$$\leq C\|h^{1/2}(Pg-U-z)\|_\Gamma, \tag{2.40}$$

where the last step is done by an inverse inequality [6]. By a similar argument as in the proof of Theorem 2.2, with the function w equal to $P(h(Pg-U-z))$ on Γ instead we get,

$$\|h^{1/2}(Pg-U-z)\|_\Gamma \leq C\epsilon\|hR(U)\|, \tag{2.41}$$

i.e.

$$\sup_{w\in H^1(\Omega)} \frac{(Pg-U-z,w-Qw)_\Gamma}{\|w\|_{1,\Omega}} \leq C\epsilon\|hR(U)\|. \tag{2.42}$$

From equation (2.24) we have $-\epsilon(f+\triangle U,Qw) = (Pg-U-z,Qw)_\Gamma$. We use this result to estimate the second term II as follows,

$$II = -\epsilon \sup_{w\in H^1(\Omega)} \frac{(f+\triangle U,Qw)}{\|w\|_{1,\Omega}} \tag{2.43}$$

$$= -\epsilon \sup_{w\in H^1(\Omega)} \frac{(-\triangle e,Qw)}{\|w\|_{1,\Omega}} \tag{2.44}$$

$$= -\epsilon \sup_{w\in H^1(\Omega)} \frac{(\nabla e,\nabla Qw)-(\partial_n e,Qw)_\Gamma}{\|w\|_{1,\Omega}} \tag{2.45}$$

$$\leq \epsilon\left(\|\nabla e\| \sup_{w\in H^1(\Omega)} \frac{\|\nabla Qw\|}{\|w\|_{1,\Omega}} + \|\partial_n e\|_{-1/2,\Gamma} \sup_{w\in H^1(\Omega)} \frac{\|Qw\|_{1/2,\Gamma}}{\|w\|_{1,\Omega}}\right). \tag{2.46}$$

From [7, 9] we know that $\|Qw\|_{1,\Omega} \leq C\|w\|_{1,\Omega}$ for locally quasi-uniform meshes. Together with the estimate, $\|Qw\|_{1/2,\Gamma} \leq C\|Qw\|_{1,\Omega}$, and equation (2.46) this gives,

$$\sup_{w\in H^1(\Omega)} \frac{(Pg-U-z,Qw)_\Gamma}{\|w\|_{1,\Omega}} \leq \epsilon C\left(\|\nabla e\| + \|\partial_n e\|_{-1/2,\Gamma}\right). \tag{2.47}$$

Equation (2.13) can now be used again for $v = e$. We get,

$$\sup_{w \in H^1(\Omega)} \frac{(Pg - U - z, Qw)_\Gamma}{\|w\|_{1,\Omega}} \leq \epsilon C \left(\|\nabla e\| + \|hR(U)\| \right). \tag{2.48}$$

Combining equation (2.33), (2.34), (2.37), and (2.48) proves the Theorem. □

Combining the estimate (2.9) of Theorem 2.1, Theorem 2.3, and the energy semi-norm estimate in Theorem 2.1 we finally end up with the following $L^2(\Omega)$-norm estimate,

$$\begin{aligned} \|e\| \leq\ & C \left(\|(h^2 + \epsilon h + \epsilon^2) R(U)\| + \|g - Pg\|_{-1/2,\Gamma} + \epsilon \|g - Pg\|_{1/2,\Gamma} \right) \\ & + \epsilon C \left(\|P(\partial_n U)\|_{-1/2,\Gamma} + \epsilon \|P(\partial_n U)\|_{1/2,\Gamma} \right). \end{aligned} \tag{2.49}$$

Remark 2.1 In the final $L^2(\Omega)$-norm estimate, equation (2.49), we see that for sufficiently smooth boundary data, g, letting $\epsilon \sim h$ would give an optimal order error for all terms but the $\epsilon C \|P(\partial_n U)\|_{-1/2,\Gamma}$ term. So if $\partial_n u \neq 0$ we need to let $\epsilon \sim h^2$ to get optimal order convergence.

3 Adaptive Strategies

We design an adaptive strategy for the energy semi-norm estimate starting from (2.8) in Theorem 2.1. Combining this result with equation (2.20) and Theorem 2.2 gives the following equation:

$$\boxed{\|\nabla e\| \leq C \left(\|hR(U)\| + \|g - Pg\|_{1/2,\Gamma} \right) + C\epsilon \left(\|P(\partial_n U)\|_{1/2,\Gamma} + \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K \right)} \tag{3.1}$$

We introduce the notation,

$$\begin{aligned} r_1 &= \|hR(U)\| + \|g - Pg\|_{1/2,\Gamma}, \\ r_2 &= \epsilon \left(\|P(\partial_n U)\|_{1/2,\Gamma} + \textstyle\sum_{K \cap \Gamma \neq \emptyset} \|R(U)\|_K \right). \end{aligned} \tag{3.2}$$

Adaptive Algorithm. The aim is to choose ϵ such that r_1 and r_2 becomes equally large.

- Let $\epsilon_0 = h$.
- Solve equation (1.2) for U.
- Calculate r_1 and r_2 according to equation (3.2).
- Determine if h-adaptivity is necessary from the size of r_1.

- Let $\epsilon = \epsilon_0 \frac{r_1}{r_2}$.

If a mesh refinement (with new mesh parameter h_{new}) was needed in step 4 we replace r_1 with $\|h_{new}R(U)\| + \|g - Pg\|_{1/2,\Gamma}$ in step 5. This procedure can then be done iteratively going from step 5 to step 2.

Remark 3.1 From experience and numerical tests for example in [13] we know that the first term in r_1 is in general over estimated due to the inequalities used to derive it. This is not the case with the other terms and this fact could be a reason to decrease ϵ even further. So even though in practice we want to use $\epsilon < \epsilon_0 r_1/r_2$ as big as possible it can we wise to choose ϵ a bit under the bound.

Remark 3.2 We can also use other norms for the adaptive strategy. One reason to choose the energy semi-norm is that $\epsilon \sim h$ since $r_1 \sim h$ and $r_2 \sim \epsilon$. If we instead consider the $L^2(\Omega)$ norm we would get $\epsilon \sim h^2$ to achieve optimal order. These results agree with earlier a priori results [6].

Remark 3.3 The main reason for not choosing ϵ too small is that the condition number of the stiffness matrix will be very large which leads to slow convergence for iterative solvers. $\epsilon \sim h$ is optimal since in this case the condition number of the matrix will not increase dramatically while for $\epsilon \sim h^2$ it will. The other reason will be illustrated in Example 3 below.

4 Numerical Examples

We present three numerical examples to verify the theoretical results of the error analysis.

Example 1. In the first example Ω is the unit square and $g = 0$ on the boundary. The load f is chosen such that the exact solution $u(x, y) = x(1-x)y(1-y)$. The aim is to use our adaptive strategy to choose ϵ in such a way that the error from the penalty method is of the same order as the discretization error. Since the exact solution is known we first present a plot, Figure 1, with the energy semi-norm of the error calculated for different h (we use quasi-uniform meshes) and ϵ. We see clearly for each h how the error eventually converges to the discretization error and we get no further improvement by decreasing ϵ.

The adaptive strategy is designed to find the biggest ϵ for which we achieve discretization error by considering the error estimators r_1 and r_2. Figure 2 shows the values of the error estimators for a fix value of $h = 0.025$. We see that the discretization part of the error r_1 is fairly constant and that the ϵ dependent part r_2 is proportional to ϵ. It is clear that the two terms r_1 and r_2 captures the essence of the behavior of the error in the energy

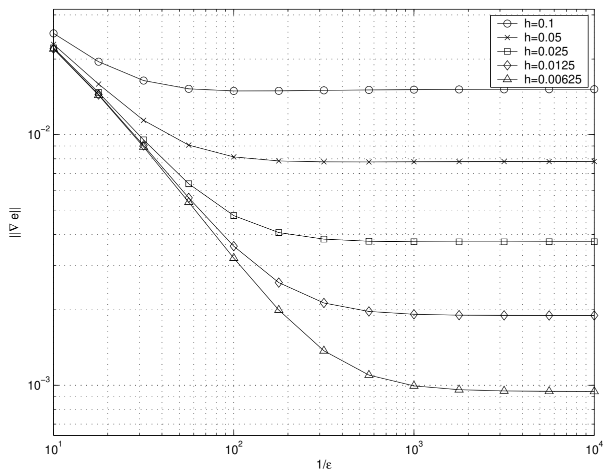

Figure 1: Error in energy semi-norm for different h and ϵ.

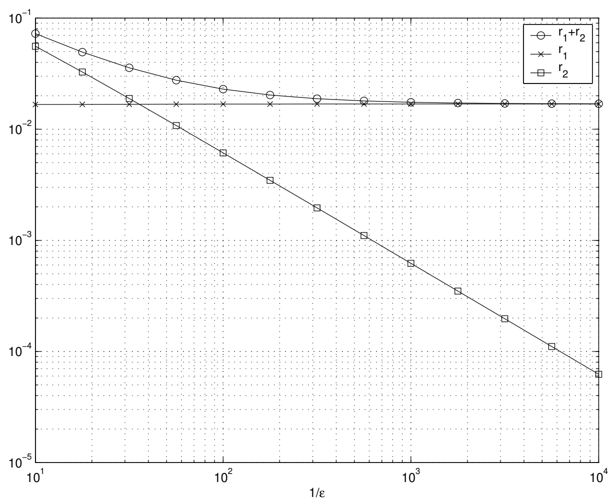

Figure 2: Error estimators dependence of ϵ.

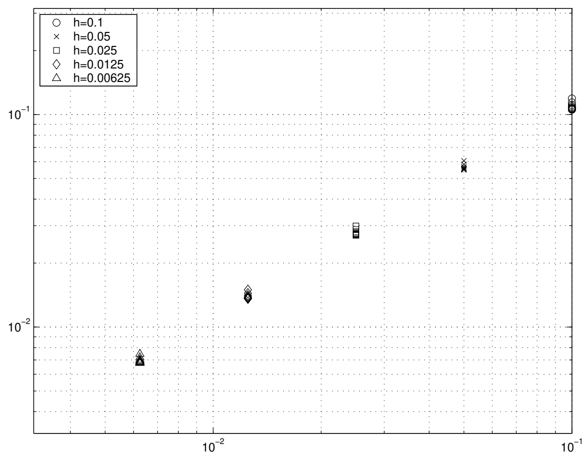

Figure 3: ϵ chosen according to adaptive strategy for different ϵ_0 and h.

semi-norm. The adaptive strategy would in this situation suggest that $\epsilon = \epsilon_0 r_1/r_2$. As seen when comparing the figures we get a slight over estimate of ϵ arising from the fact that r_1 is over estimated.

To sum up this example we analyze the h-dependence of ϵ in our method. In this particular example $\epsilon = \epsilon_0 r_1/r_2$ for different ϵ_0 in the range 10^{-1} to 10^{-7}. As seen from the small clusters in Figure 3 we get very similar results on ϵ for different ϵ_0. We also recognize that ϵ is proportional to h.

Example 2. Next we turn our attention to a situation where $g \notin V$ on one part of the boundary. We let $g = 0$ on three parts of the unit square and on the fourth part we let g be saw shaped as seen in Figure 4. The peaks and valleys are chosen so that they do not coincide with the mesh. Using a constant ϵ would in this example not be the best approach since we need a very small ϵ just on a part of the boundary where the normal derivative of the solution is large. Motivated by the results in Theorem 2.2 we use two different values of ϵ, ϵ_1 on the simple part and ϵ_2 on the complicated part. In Figure 5 we see the result of using our algorithm with $\epsilon_0 = h$ as a starting guess for different h. The penalty parameter is chosen as

$$\epsilon_i = \epsilon_0 \frac{|\Gamma_i|}{|\Gamma|} \frac{\|hR(U)\|}{\|g - U\|_{1/2,\Gamma_i}}, \tag{4.1}$$

where $|\Gamma_i|$ is the length of the boundary segment Γ_i. If the function g allows it can be convenient to replace $\|g - U\|_{1/2,\Gamma}$ by $\|g - U\|_{\Gamma} \|g - U\|_{1,\Gamma}$ in practice. This gives a lower value of ϵ but is simpler to compute. It is clear that the algorithm suggests us to choose a much higher ϵ on the simple part of the domain. We also see that both ϵ_1 and ϵ_2 are proportional to h just with different constants.

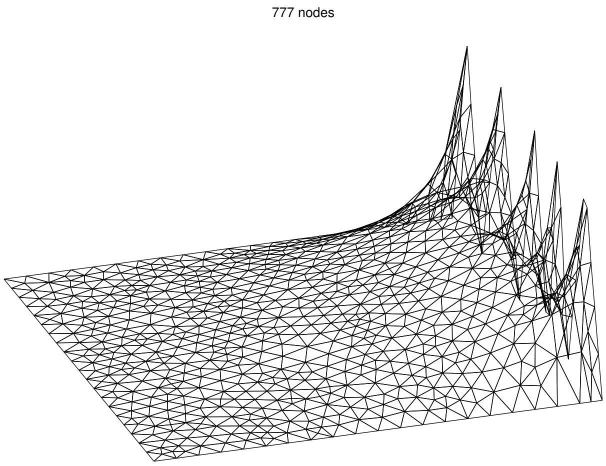

Figure 4: Solution to the second test problem

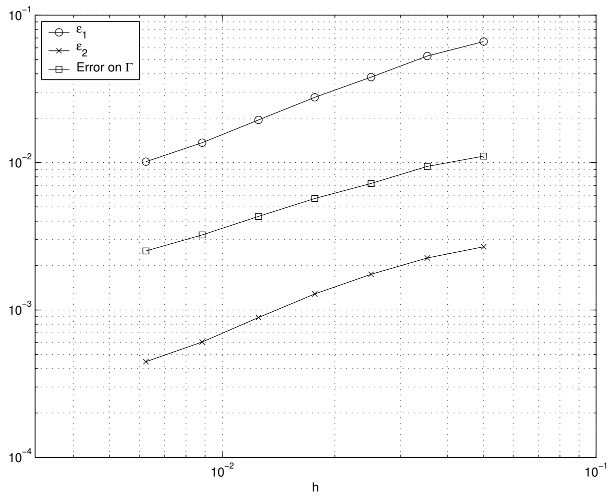

Figure 5: The boundary error $\|e\|_\Gamma$ and ϵ_i calculated for different values of h.

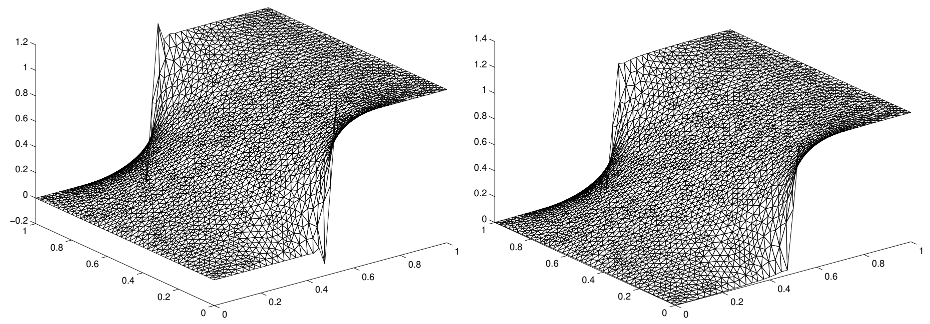

Figure 6: 10 times the optimal ϵ to the left and optimal ϵ to the right.

Example 3. Finally we study an interesting effect that can arise from choosing ϵ to small. From the earlier a priori work [3, 4] it is clear that this can lead to problems. This effect can not be seen explicitly from the a posteriori error estimates but it can be taken care of using the proposed adaptive strategy.

We let g be close to discontinuous, zero on one part of the boundary and one on the other with a very steep sloop that connects the parts, see Figure 6. Further we let $f = 1$. We solve the problem by iterating the adaptive algorithm starting from $\epsilon = h = 1/40$ and find an optimal $\epsilon = 1/151$, see Figure 6 (right). Then we solve the same problem using a ten times smaller $\epsilon = 1/1510$ (left). We see clearly that a too small choice of ϵ for this problem leads to oscillations in the solution. If ϵ is decreased further the effect is even stronger.

The reason for this behavior is that equation (1.2) will force $U \approx Pg$ if ϵ is very small and it is known that the L^2 projection P has oscillating behavior for discontinuous data. This example together with the size of the condition number motivates using the adaptive procedure when choosing ϵ.

5 Conclusion

We have derived two a posteriori error estimates and designed an adaptive strategy for choosing the penalty parameter ϵ in BPM for one of these. We present numerical examples that confirms our theoretical results and we conclude that by this strategy we achieve optimal order convergence for piecewise linears which agrees with earlier a priori work.

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