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STABILIZED LAGRANGE MULTIPLIER METHODS FOR ELASTIC CONTACT WITH FRICTION

PER HEINTZ AND PETER HANSBO

Abstract. In most finite element (FE) codes contact is checked only at the nodes, corresponding to the use of pointwise constraints. However, this approach might not be stable in case the bodies coming into contact have non-matching grids at the contact interface. To alleviate this problem, we propose a stabilized Lagrange multiplier method, based on a global polynomial multiplier, for the finite element solution of (non)linear elastic contact problems with non-matching grids. In particular, our approach allows us to avoid integrating products of different finite element basis functions on the surface meshes at the contact zone.

1. Introduction

One basic question in contact problems involving non-matching grids is how to handle the quadrature problem on the contact interface. The Lagrange multiplier space is, typically, defined as piecewise polynomials on one side of the bodies that are coming into contact. Thus, it is necessary to integrate basis functions defined on two different surface grids which leads to an expensive search problem in locating quadrature points on the boundary of the neighboring grid. Another issue is the number of degrees of freedom in the discrete Lagrange multiplier space. If too many constraints are used, the discrete system might be singular. On the other hand, if there are too few constraints, there might be unphysical violation of the non-penetration condition. There are, basically, two different possibilities to obtain a stable discretization. The first approach is to choose discrete spaces that fulfill the *inf–sup* condition which guarantees stability (cf. [7]). A well-known example of such a scheme is the mortar method (which can be given a Lagrange multiplier interpretation), see [4]. A drawback of the mortar method is that the end points of the contact zone (in two dimensions) should match, i.e., be nodes on both meshes. The other option is to change the bilinear form in such a way that stability is ensured, and this is the approach taken in this paper.

We extend the stabilized Lagrange multiplier method based on a global polynomial discretization of the multiplier space introduced by Hansbo, Lovadina, Perugia, and Sangalli [10] to the case of elastic contact, focusing on a symmetric formulation. Unlike previous

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stabilization methods proposed in the literature, e.g., [5, 8], as well as the mortar method, we avoid the integration of products of finite element basis functions on the different surface meshes.

Although the subsequent material is focused on contact mechanics, a similar approach could be used for more general interface problems where we do not want to enforce matching grids at the interface. The rest of the paper is organized as follows. First, we describe the proposed method. Secondly, we make some comments on the discretization and finally, we present some numerical results.

2. Problem formulation

Let Ω_i, $i = 1, 2$, be two bounded domains in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma_i = \Gamma_i^{\mathrm{D}} \cup \Gamma_i^{\mathrm{N}} \cup \Gamma^{\mathrm{C}}$. We consider the case where the domains are subjected to proper Dirichlet and Neumann boundary conditions and are coming into frictionless contact along an interface Γ^{C}. Thus, we want to solve the problem

$$\left.\begin{array}{rcll} -\nabla\cdot{\boldsymbol{\sigma}_i}^{\mathrm{T}} & = & \boldsymbol{f}_i & \text{in } \Omega_i, \\ \boldsymbol{u}_i & = & \boldsymbol{g}_i & \text{on } \Gamma_i^{\mathrm{D}}, \\ \boldsymbol{\sigma}_i\cdot\boldsymbol{n}_i & = & \boldsymbol{t}_i & \text{on } \Gamma_i^{\mathrm{N}}, \\ [\![\boldsymbol{u}]\!]\cdot\boldsymbol{n} & = & 0 & \text{on } \Gamma^{\mathrm{C}}, \\ [\![\boldsymbol{\sigma}\cdot\boldsymbol{n}]\!]\cdot\boldsymbol{n} & = & 0 & \text{on } \Gamma^{\mathrm{C}}, \\ \boldsymbol{\sigma}_i & = & \multicolumn{2}{l}{\dfrac{E}{1+\nu}\boldsymbol{\varepsilon}(\boldsymbol{u}_i) + \dfrac{\nu E}{(1+\nu)(1-2\nu)}\nabla\cdot\boldsymbol{u}_i\boldsymbol{I},} \end{array}\right\} \tag{2.1}$$

where $\boldsymbol{f}_i$, $\boldsymbol{g}_i$ and $\boldsymbol{t}_i$ are the body load, prescribed displacements and tractions respectively. $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the (small) strain tensor with components $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u}\otimes\nabla + \nabla\otimes\boldsymbol{u})$, where $(\boldsymbol{w}\otimes\boldsymbol{v})_{ij} = w_i v_j$, E is the modulus of elasticity, ν is Poisson's ratio, and $\boldsymbol{I}$ is the identity tensor. We shall assume that E and ν are constant in each domain, but possibly different between domains. The extension to varying material parameters does not present any serious difficulties. Concerning the interface conditions, $[\![\cdot]\!] := (\cdot_1 - \cdot_2)$ denotes the jump in the argument across the interface where $\boldsymbol{n} := \boldsymbol{n}_1 = -\boldsymbol{n}_2$ is the outward pointing normal on the boundary of Ω_1 (master surface). The relation $[\![\boldsymbol{u}]\!]\cdot\boldsymbol{n} = 0$ corresponds to the non-penetration condition and $[\![\boldsymbol{\sigma}\cdot\boldsymbol{n}]\!]\cdot\boldsymbol{n} = 0$ corresponds to equilibrium in normal traction across the interface. In the following, we shall use the notation $\sigma_n := \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}$ for the normal stress on Γ^{C}.

We shall in the following assume that the contact zone Γ^{C} is known. The method for the case of varying contact zone follows easily and is explained in the section on implementation.

A weak form of (2.1), using the Lagrange multiplier method, can be formulated as follows: find

$$\boldsymbol{u} \in \vec{V}_g := \{\boldsymbol{v}_i : \boldsymbol{v}_i \in \left[H^1(\Omega_i)\right]^d, \boldsymbol{v}_i = \boldsymbol{g}_i \text{ on } \Gamma_i^{\mathrm{D}}\}$$

and $\lambda \in H^{-1/2}(\Gamma^{\mathrm{C}})$ such that

$$B(\boldsymbol{u}, \lambda, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \vec{V}_0 \tag{2.2}$$

$$C(\boldsymbol{u}, \mu) = 0 \qquad \forall \mu \in H^{-1/2}(\Gamma^{\mathrm{C}}) \tag{2.3}$$

where

$$B(\boldsymbol{u},\lambda,\boldsymbol{v}) = \sum_i \left(\int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i) : \boldsymbol{\varepsilon}(\boldsymbol{v}_i)\,dx\right) + \int_{\Gamma^{\mathrm{C}}} \lambda\,[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]\,ds,$$

$$L(\boldsymbol{v}) = \sum_i \left(\int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i\,dx\right) + \sum_i \int_{\Gamma_i^{\mathrm{N}}} \boldsymbol{t}_i \cdot \boldsymbol{v}_i\,ds,$$

and

$$C(\boldsymbol{u},\mu) = \int_{\Gamma^{\mathrm{C}}} [\![\boldsymbol{u}\cdot\boldsymbol{n}]\!]\,\mu\,ds.$$

This formulation forms the starting point for our finite element approximation.

3. Finite element methods

Assume that we are given triangular meshes $\mathcal{T}_i^h$ of the domains Ω_i, $i = 1, 2$. We denote by h_i the meshsize of $\mathcal{T}_i^h$. Obviously, $\mathcal{T}^h = \mathcal{T}_1^h \cup \mathcal{T}_2^h$ provides a mesh for Ω, whose mesh-size is $h = \max\{h_1, h_2\}$. We introduce the (family of) finite element spaces

$$\vec{V}_g^h = \{\boldsymbol{v}_i : \boldsymbol{v}_i \in \left[H^1(\Omega_i)\right]^d,:\ \boldsymbol{v}_i|_K \in [P^k(K)]^d,\ \forall K \in \mathcal{T}^h,\ \boldsymbol{v}_i = \pi_h \boldsymbol{g} \,\mathrm{on}\, \Gamma_i^{\mathrm{D}}\},$$

where $P^k(K)$ denotes the space of polynomials of degree at most k on K, with $k \geq 1$ and π_h is the nodal interpolant. On Γ^{C} we introduce a family of spaces Λ^p of discrete multipliers. As a particular case, we will consider the space Λ^p of global polynomials defined as follows: the interface Γ^{C} is decomposed as the union $\Gamma = \bigcup \Gamma_j^{\mathrm{C}}$ of $n_{\Gamma^{\mathrm{C}}}$ straight lines Γ_j^{C} of length ℓ_j; we associate with each Γ_j^{C} the non-negative integer p_j and define $\underline{p} := [p_1, \ldots, p_{n_{\Gamma^{\mathrm{C}}}}]$; then our particular choice is

$$\Lambda^p = \{\mu \in \Lambda :\ \mu|_{\Gamma_j^{\mathrm{C}}} \in P^{p_j}(\Gamma_j^{\mathrm{C}}),\ j = 1, \ldots, n_{\Gamma^{\mathrm{C}}}\}, \tag{3.1}$$

with $P^{p_j}(\Gamma_j^{\mathrm{C}})$ denoting the space of polynomials of degree at most p_j on Γ_j^{C} with respect to a local coordinate. In this particular case the elements of Λ^p can be discontinuous at the endpoints of the Γ_j^{C}'s.

Now, when using stable Lagrange multiplier methods with locally defined multipliers on one of the trace meshes (or associated projections, cf. [4]), as well as when using most stabilized methods (e.g., [8, 5]) there still remains the practical problem of how to evaluate integrals of the type

$$\int_{\Gamma^{\mathrm{C}}} \boldsymbol{v}_i \cdot \boldsymbol{n}\,\mu\,ds, \tag{3.2}$$

where the discrete space for $\boldsymbol{v}_i$ and μ are defined on different grids. If quadrature is used, we have an expensive search problem in locating elements on one side of the interface containing quadrature points on the other side of the interface. Using globally defined polynomials allows us to take an alternative route that simplifies the implementation.

3.1. **An inconsistent penalty method.** In order to handle the quadrature on the interface, we consider first a perturbed Lagrangian method and seek $\boldsymbol{u}^h \in \vec{V}_g^h$ and $\lambda^p \in \Lambda^p$ such that

$$B(\boldsymbol{u}^h, \lambda^p, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \vec{V}_h^0 \tag{3.3}$$

$$C_{\mathrm{P}}(\boldsymbol{u}^h, \mu) = 0 \qquad \forall \mu \in \Lambda^p \tag{3.4}$$

where

$$C_{\mathrm{P}}(\boldsymbol{u}, \mu) := \int_{\Gamma^{\mathrm{C}}} [\![\boldsymbol{u} \cdot \boldsymbol{n}]\!]\, \mu \, ds - \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma} \lambda \mu \, ds.$$

Now, as $\gamma \to \infty$ this will coincide with a standard Lagrange multiplier method which requires balancing between the discrete spaces for λ and $\boldsymbol{u}$ which have to fulfill a BB condition. It is possible to let $\gamma = Ch^{-\alpha}$ such that the problem of balancing the discrete spaces is alleviated, see Barret and Elliot [6]. Furthermore, the product of basis functions and global polynomials can (at least for simplicial elements) easily be integrated exactly. However, for optimal convergence, the number α must be tied to the polynomial approximation used in $\vec{V}^h$, affecting the condition number of the resulting system matrix.

3.2. **Consistent methods.** In [8], Becker, Hansbo and Stenberg extended the classical method of Nitsche [11] for handling Dirichlet boundary conditions to include domain decomposition with non-matching grids. The problem of having to integrate products of functions on one side of the interface with functions on the other is present also in their method. However, Nitsche-type methods are consistent and thus optimally convergent with a mesh dependent penalty parameter of $O(h^{-1})$. We wish to retain the optimality of convergence with a fixed penalty–like parameter while removing the need to integrate products of functions on unrelated meshes. Using the notation

$$\{\boldsymbol{u}\} := \alpha \boldsymbol{u}_1 + (1-\alpha)\boldsymbol{u}_2, \quad \alpha \in [0,1]\,, \tag{3.5}$$

we therefore propose the following method: find $\boldsymbol{u}^h \in \vec{V}_g^h$ and $\lambda^p \in \Lambda^p$ such that

$$B(\boldsymbol{u}^h, \lambda^p, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \vec{V}_0^h \tag{3.6}$$

$$C_{\mathrm{C}}(\boldsymbol{u}^h, \mu) = 0 \qquad \forall \mu \in \Lambda^p \tag{3.7}$$

where

$$C_{\mathrm{C}}(\boldsymbol{u}, \mu) := \int_{\Gamma^{\mathrm{C}}} [\![\boldsymbol{u} \cdot \boldsymbol{n}]\!]\, \mu \, ds - \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma} \{\sigma_n(\boldsymbol{u})\} \mu \, ds - \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma} \lambda \mu \, ds.$$

This is a consistent method: the Galerkin orthogonality gives, since formally $\lambda = -\sigma_n(\boldsymbol{u})$, that

$$\begin{aligned} B(\boldsymbol{u} - \boldsymbol{u}^h, \lambda - \lambda^p, \boldsymbol{v}) &= 0 \qquad \forall \boldsymbol{v} \in \vec{V}_0^h \\ C_{\mathrm{C}}(\boldsymbol{u} - \boldsymbol{u}_h, \mu) &= 0 \qquad \forall \mu \in P^k(\Gamma^{\mathrm{C}}) \end{aligned}$$

The resulting matrix system will however not be symmetric. In order to restore symmetry, we follow [10] and add further interface terms. We seek $\boldsymbol{u}^h \in \vec{V}_g$ and $\lambda^p \in \Lambda^p$ such that

$$B_S(\boldsymbol{u}^h, \lambda^p, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \vec{V}_0^h \tag{3.8}$$

$$C_C(\boldsymbol{u}^h, \mu) = 0 \qquad \forall \mu \in \Lambda^p \tag{3.9}$$

where

$$\begin{aligned} B_S(\boldsymbol{u}, \lambda, \boldsymbol{v}) &:= \sum_i \left(\int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i) : \boldsymbol{\varepsilon}(\boldsymbol{v}_i)\, dx \right) + \int_{\Gamma^C} \lambda \, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] \, ds \\ &\quad - \int_{\Gamma^C} \frac{1}{\gamma} \lambda \{\sigma_n(\boldsymbol{v})\}\, ds - \int_{\Gamma^C} \frac{1}{\gamma} \{\sigma_n(\boldsymbol{u})\}\{\sigma_n(\boldsymbol{v})\}\, ds. \end{aligned}$$

Note that as long as $\alpha = 0$ or $\alpha = 1$, the integration of cross terms on the interface is still avoided.

4. Stability of the consistent symmetric method

In this section, we will show that our method is stable for any combination of approximating spaces $\vec{V}^h$ and Λ^p, on condition that γ is chosen appropriately. For simplicity, we will assume that $\boldsymbol{g}_i = 0$ and that Γ_i^D has nonzero measure for $i = 1, 2$, so that Korn's inequality holds. The problem of "floaters", e.g., Ω_1 enclosed by Ω_2 or vice versa, can be dealt with following [10]. For definiteness, we assume that $\alpha = 1$ in the following, and thus we have $\{\sigma_n(\boldsymbol{u})\} = \sigma_n(\boldsymbol{u}_1)$. We define the mesh-dependent parameter γ as follows. Set $\kappa := E/(1-2\nu)$ and define

$$\gamma := \gamma_0 \, \kappa_1 h_{K_1}^{-1},$$

with γ_0 constant independent on the mesh-size and the material properties, to be further specified below. For simplicity, we will also assume that the meshsize $h := h_{K_1}$ is constant on the surface mesh of Ω_1.

As a shorthand notation, we define

$$\begin{aligned} \mathcal{B}^h(\boldsymbol{u}, \lambda; \boldsymbol{v}, \mu) &:= \sum_i \int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i) : \boldsymbol{\varepsilon}(\boldsymbol{v}_i)\, dx + \int_{\Gamma^C} \lambda \, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] \, ds - \int_{\Gamma^C} [\![\boldsymbol{u} \cdot \boldsymbol{n}]\!] \, \mu \, ds \\ &\quad + \int_{\Gamma^C} \frac{1}{\gamma} \{\sigma_n(\boldsymbol{u})\} \mu \, ds + \int_{\Gamma^C} \frac{1}{\gamma} \lambda \, \mu \, ds \\ &\quad - \int_{\Gamma^C} \frac{1}{\gamma} \lambda \{\sigma_n(\boldsymbol{v})\}\, ds - \int_{\Gamma^C} \frac{1}{\gamma} \{\sigma_n(\boldsymbol{u})\}\{\sigma_n(\boldsymbol{v})\}\, ds, \end{aligned}$$

and we have the following consistency relation:

$$\mathcal{B}^h(\boldsymbol{u} - \boldsymbol{u}^h, \lambda - \lambda^p; \boldsymbol{v}, \mu) = 0, \tag{4.1}$$

for all $\boldsymbol{v} \in \vec{V}_0^h$ and $\mu \in \Lambda^p$.

Introducing the norms

$$\|\boldsymbol{u}\|_B := \left(\sum_i \int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i) : \boldsymbol{\varepsilon}(\boldsymbol{u}_i)\, dx \right)^{1/2}$$

and

$$|||(\boldsymbol{u},\lambda)||| := \left(\|\boldsymbol{u}\|_B^2 + \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma}\,\lambda^2\,ds\right)^{1/2},$$

we can also show coercivity of the discrete problem with respect to $|||\cdot|||$, in the following sense.

PROPOSITION There holds the *inf-sup* condition

$$\sup_{(\boldsymbol{v},\mu)\in\vec{V}_0^h\times\Lambda^p} \frac{\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;\boldsymbol{v},\mu)}{|||(\boldsymbol{v},\mu)|||} \geq C\,|||\,(\boldsymbol{u}^h,\lambda^p)|||,\quad \forall(\boldsymbol{u}^h,\lambda^p)\in\vec{V}_0^h\times\Lambda^p, \tag{4.2}$$

for some C independent of the meshsize.

PROOF The proof follows the lines of [10] where the corresponding result for the unsymmetric method applied to Poisson's equation was shown. We make the particular choice $\boldsymbol{v} = \boldsymbol{u}^h$ and $\mu = \mu_1 + \delta\mu_2$, with $\mu_1 = \lambda^p$ and $\mu_2 = -h^{-1}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]$, where π_p is the $L_2(\Gamma^{\mathrm{C}})$–projection operator onto Λ^p and δ is a free positive parameter to be determined. Then we have

$$\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;\boldsymbol{v},\mu_1) = \|\boldsymbol{u}^h\|_B^2 - \|\gamma^{-1/2}\{\sigma_n(\boldsymbol{u}^h)\}\|_{L_2(\Gamma^{\mathrm{C}})}^2 + \|\gamma^{-1/2}\lambda^p\|_{L_2(\Gamma^{\mathrm{C}})}^2,$$

and by the inverse inequality (see [9])

$$\|\kappa_1^{-1/2}h^{1/2}\sigma_n(\boldsymbol{w}_1)\|_{L_2(\Gamma^{\mathrm{C}})}^2 \leq C_I\|\boldsymbol{w}\|_B^2,\qquad \forall\boldsymbol{w}\in\vec{V}_0^h, \tag{4.3}$$

we obtain, provided that $\gamma_0 > C_I$,

$$\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;\boldsymbol{v},\mu_1) \geq C\,|||\,(\boldsymbol{u}^h,\lambda^p)|||^2 \tag{4.4}$$

where C depends on γ_0 and C_I but is independent of the mesh-size.

Next, we see that

$$\begin{aligned}
\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;0,\mu_2) &= \int_{\Gamma^{\mathrm{C}}} \frac{[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]}{h}\,\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\,ds - \int_{\Gamma^{\mathrm{C}}} \frac{\{\sigma_n(\boldsymbol{u}^h)\}}{\gamma\,h}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\,ds \\
&\quad - \int_{\Gamma^{\mathrm{C}}} \frac{\lambda^p}{\gamma\,h}\,\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\,ds \\
&= \|h^{-1/2}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\|_{L_2(\Gamma^{\mathrm{C}})}^2 - \int_{\Gamma^{\mathrm{C}}} \frac{\{\sigma_n(\boldsymbol{u}^h)\}}{\gamma\,h}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\,ds \\
&\quad - \int_{\Gamma^{\mathrm{C}}} \frac{\lambda^p}{\gamma\,h}\,\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\,ds.
\end{aligned}$$

Since

$$\int_{\Gamma^{\mathrm{C}}} \frac{\{\sigma_n(\boldsymbol{u}^h)\}}{\gamma\,h}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\,ds \leq \frac{\|\kappa_1^{-1/2}h^{1/2}\{\sigma_n(\boldsymbol{u}^h)\}\|_{L_2(\Gamma^{\mathrm{C}})}}{\gamma_0\kappa_1^{1/2}}\|h^{-1/2}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\|_{L_2(\Gamma^{\mathrm{C}})}$$

and

$$\int_{\Gamma^{\mathrm{C}}} \frac{\lambda^p}{\gamma\,h}\,\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\,ds \leq \frac{\|\gamma^{-1/2}\lambda^p\|_{L_2(\Gamma^{\mathrm{C}})}}{\gamma_0^{1/2}\kappa_1^{1/2}}\|h^{-1/2}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\|_{L_2(\Gamma^{\mathrm{C}})}$$

we can use the arithmetic-geometric mean inequality for the last two integrals, followed by the inverse inequality (4.3), to obtain

$$
\begin{aligned}
\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;0,\mu_2) \;\geq\; & \frac{1}{2}\|h^{-1/2}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\|^2_{L_2(\Gamma^{\mathrm{C}})}-\frac{C_I}{\kappa_1\gamma_0^2}\|\boldsymbol{u}^h\|^2_B \\
& -\frac{1}{\kappa_1\gamma_0}\|\gamma^{-1/2}\lambda^p\|^2_{L_2(\Gamma^{\mathrm{C}})}.
\end{aligned}
\tag{4.5}
$$

By adding together (4.4) and (4.5) multiplied by δ, taking δ small enough, we obtain

$$
\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;\boldsymbol{v},\mu)\geq c\left(|\!|\!|(\boldsymbol{u}^h,\lambda^p)\,|\!|\!|^2+\|h^{-1/2}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\|^2_{L_2(\Gamma^{\mathrm{C}})}\right),
$$

with a positive constant c only depending on κ_1, C_I, and γ_0, therefore independent of the mesh-size. Consequently,

$$
\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;\boldsymbol{v},\mu)\geq C_0\,|\!|\!|\,(\boldsymbol{u}^h,\lambda^p)\,|\!|\!|^2\,. \tag{4.6}
$$

Finally, by a trace inequality and Korn's inequality,

$$
\|\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!]\|_{L_2(\Gamma^{\mathrm{C}})}\leq C\|\boldsymbol{u}^h\|_B,
$$

with C depending on Ω_i, $i=1,2$, so that

$$
|\!|\!|(\boldsymbol{v},\mu)|\!|\!|=|\!|\!|(\boldsymbol{u}^h,\lambda^p-h^{-1}\pi_p[\![\boldsymbol{u}^h\cdot\boldsymbol{n}]\!])|\!|\!|\leq C_1\,|\!|\!|\,(\boldsymbol{u}^h,\lambda^p)\,|\!|\!|\,. \tag{4.7}
$$

Thus, from (4.6) and (4.7),

$$
\frac{\mathcal{B}^h(\boldsymbol{u}^h,\lambda^p;\boldsymbol{v},\mu)}{|\!|\!|(\boldsymbol{v},\mu)|\!|\!|}\geq\frac{C_0}{C_1}\,|\!|\!|\,(\boldsymbol{u}^h,\lambda^p)|\!|\!|
$$

which proves the proposition.
We refer to [10] for convergence proofs, using the stability and consistency, in a setting similar to ours.

5. Incorporating a Coulomb friction law

We will also give some details about our implementation of a simple Coulomb friction law in the present setting.

Consider the tangential stress vector $\boldsymbol{\sigma}_t:=\boldsymbol{\sigma}\cdot\boldsymbol{n}-\sigma_n\boldsymbol{n}$. A simple friction model, cf. Wriggers [13], is to define the elastic zone (of no slip) as

$$
\mathbb{E}=\{\boldsymbol{\sigma}_t:\;\|\boldsymbol{\sigma}_t\|-\mu\sigma_n<0\}\,,
$$

where $\|\cdot\|$ is the Euclidean vector norm and μ is the friction coefficient, assumed known and constant. When $\|\boldsymbol{\sigma}_t\|=\mu\sigma_n$, we have slip and the magnitude of the tangential stress vector is given implicitly. Thus, in a nonlinear iteration scheme, we can use the simple approach of scaling the tangential stress. Splitting the contact boundary into a slip part, $\Gamma^{\mathrm{C}}_{\mathrm{S}}$, and a no-slip part, $\Gamma^{\mathrm{C}}_{\mathrm{N}}$, so that $\Gamma^{\mathrm{C}}=\Gamma^{\mathrm{C}}_{\mathrm{S}}\cup\Gamma^{\mathrm{C}}_{\mathrm{N}}$ and $\Gamma^{\mathrm{C}}_{\mathrm{S}}\cap\Gamma^{\mathrm{C}}_{\mathrm{N}}=\emptyset$, and introducing a vector-valued multiplier, we seek $\boldsymbol{u}^h\in\vec{V}^h_g$ and $\boldsymbol{\lambda}^p\in[\Lambda^p]^d$ such that

$$\sum_i \left(\int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i^h) : \boldsymbol{\varepsilon}(\boldsymbol{v}_i)\, dx \right) + \int_{\Gamma_{\mathrm{N}}^{\mathrm{C}}} \boldsymbol{\lambda}^p \cdot [\![\boldsymbol{v}]\!]\, ds - \int_{\Gamma_{\mathrm{N}}^{\mathrm{C}}} \frac{1}{\gamma} \boldsymbol{\lambda}^p \cdot \{\boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\}\, ds$$
$$- \int_{\Gamma_{\mathrm{N}}^{\mathrm{C}}} \frac{1}{\gamma} \{\boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}^h)\}\{\boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v})\}\, ds + \int_{\Gamma_{\mathrm{S}}^{\mathrm{C}}} \boldsymbol{n} \cdot \boldsymbol{\lambda}^p [\![\boldsymbol{n} \cdot \boldsymbol{v}]\!]\, ds - \int_{\Gamma_{\mathrm{S}}^{\mathrm{C}}} \frac{1}{\gamma} \boldsymbol{n} \cdot \boldsymbol{\lambda}^p \{\sigma_n(\boldsymbol{v})\}\, ds$$
$$- \int_{\Gamma_{\mathrm{S}}^{\mathrm{C}}} \frac{1}{\gamma} \{\sigma_n(\boldsymbol{u}^h)\}\{\sigma_n(\boldsymbol{v})\}\, ds = L(\boldsymbol{v}) + \int_{\Gamma_{\mathrm{S}}^{\mathrm{C}}} \boldsymbol{t}^* \cdot [\![(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{v}]\!]\, ds$$

for all $\boldsymbol{v} \in \vec{V}_0^h$, and

$$\int_{\Gamma_{\mathrm{N}}^{\mathrm{C}}} [\![\boldsymbol{u}^h]\!] \cdot \boldsymbol{\mu}\, ds - \int_{\Gamma_{\mathrm{N}}^{\mathrm{C}}} \frac{1}{\gamma} \{\boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}^h)\} \cdot \boldsymbol{\mu}\, ds - \int_{\Gamma_{\mathrm{N}}^{\mathrm{C}}} \frac{1}{\gamma} \boldsymbol{\lambda}^p \cdot \boldsymbol{\mu}\, ds$$
$$+ \int_{\Gamma_{\mathrm{S}}^{\mathrm{C}}} [\![\boldsymbol{n} \cdot \boldsymbol{u}^h]\!]\, \boldsymbol{n} \cdot \boldsymbol{\mu}\, ds - \int_{\Gamma_{\mathrm{S}}^{\mathrm{C}}} \frac{1}{\gamma} \{\sigma_n(\boldsymbol{u}^h)\} \boldsymbol{n} \cdot \boldsymbol{\mu}\, ds - \int_{\Gamma_{\mathrm{S}}^{\mathrm{C}}} \frac{1}{\gamma} \boldsymbol{n} \cdot \boldsymbol{\lambda}^p \boldsymbol{n} \cdot \boldsymbol{\mu}\, ds = 0$$

for all $\boldsymbol{\mu} \in [\Lambda^p]^d$. Defining the discrete tangential and normal stresses as

$$\boldsymbol{\sigma}_t^p := -(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{\lambda}^p, \quad \sigma_n^p := -\boldsymbol{n} \cdot \boldsymbol{\lambda}^p,$$

we denote by $\boldsymbol{t}^*$the closest point projection of the (trial) stress onto the surface

$$\partial \mathbb{E}^p = \{\boldsymbol{\sigma}_t^p : \ \|\boldsymbol{\sigma}_t^p\| - \mu \sigma_n^p = 0\}.$$

6. Discretization - Implementation

In order to achieve as good accuracy as possible for a given grid, the number of Lagrange multipliers should, in general, be of the order max number of nodes on one side of the interface (if the solution is smooth enough, it was however shown in [10] that the number of multipliers can be chosen significantly smaller). With stabilization one can use even more multipliers, but the method then becomes unnecessarily expensive.

An advantage of the present method, using global polynomials for λ, is that the number of active Lagrange multipliers is constant between each iteration. It is not necessary to add or remove rows and columns in the system matrix in case the contact zone shrinks or expands over element boundaries.

We also remark that it is possible to decouple the equation system such that it can be solved in parallel, as in [12].

6.1. **The symmetric and consistent coefficient matrix - with and without friction.** The discrete coefficient matrix for the symmetric, consistent method with or without friction will have the following structure

$$\begin{bmatrix} \boldsymbol{B}_1 + \alpha^2 \boldsymbol{S}_{11} & \alpha(1-\alpha)\boldsymbol{S}_{12} & \boldsymbol{C}_1 + \alpha \boldsymbol{S}_1 \\ \alpha(1-\alpha)\boldsymbol{S}_{21} & \boldsymbol{B}_2 + (1-\alpha)^2 \boldsymbol{S}_{22} & \boldsymbol{C}_2 + (1-\alpha)\boldsymbol{S}_2 \\ \boldsymbol{C}_1^{\mathrm{T}} + \alpha \boldsymbol{S}_1^{\mathrm{T}} & \boldsymbol{C}_2^{\mathrm{T}} + (1-\alpha)\boldsymbol{S}_2^{\mathrm{T}} & \boldsymbol{S} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \Lambda \end{bmatrix}, \tag{6.1}$$

where the subscript refers to the involved bodies in the integrands. In particular, if we choose $\alpha = 0$ or $\alpha = 1$, *i.*e., the corresponding terms are evaluated on one side of the interface, we do not have to integrate basis functions defined on different sides of the interface. In (6.1), for the non-friction case, the submatrices correspond to the terms in the weak formulation as follows.

$$\boldsymbol{B}_i \longleftrightarrow \int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i^h) : \boldsymbol{\varepsilon}(\boldsymbol{v}_i)\, dx \tag{6.2}$$

$$\boldsymbol{C}_i \longleftrightarrow \int_{\Gamma^{\mathrm{C}}} \lambda^p\, \boldsymbol{v}_i \cdot \boldsymbol{n}\, ds, \qquad \boldsymbol{C}_i^{\mathrm{T}} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}} \boldsymbol{u}_i^h \cdot \boldsymbol{n}\, \mu\, ds \tag{6.3}$$

$$\boldsymbol{S} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma} \lambda^p \mu\, ds \tag{6.4}$$

$$\boldsymbol{S}_i \longleftrightarrow \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma} \lambda^p \sigma_n(\boldsymbol{v}_i)\, ds, \qquad \boldsymbol{S}_i^{\mathrm{T}} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma} \sigma_n(\boldsymbol{u}_i^h) \mu\, ds \tag{6.5}$$

$$\boldsymbol{S}_{ij} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}} \frac{1}{\gamma} \sigma_n(\boldsymbol{u}_i^h) \cdot \sigma_n(\boldsymbol{v}_j)\, ds. \tag{6.6}$$

Incorporating the Coulomb friction formulation, using vector valued multipliers, the contact zone will be subdivided into a stick region, $\Gamma^{\mathrm{C}}_{\mathrm{S}}$, and a non-stick region, $\Gamma^{\mathrm{C}}_{\mathrm{N}}$, and the matrices will correspond to the weak terms as follows.

$$\boldsymbol{B}_i \longleftrightarrow \int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i^h) : \boldsymbol{\varepsilon}(\boldsymbol{v}_i)\, dx \tag{6.7}$$

$$\boldsymbol{C}_i \longleftrightarrow \int_{\Gamma^{\mathrm{C}}_{\mathrm{S}}} \boldsymbol{\lambda}^p \cdot \boldsymbol{v}_i\, ds + \int_{\Gamma^{\mathrm{C}}_{\mathrm{N}}} (\boldsymbol{\lambda}^p \cdot \boldsymbol{n})(\boldsymbol{v}_i \cdot \boldsymbol{n})\, ds \tag{6.8}$$

$$\boldsymbol{C}_i^{\mathrm{T}} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}_{\mathrm{S}}} \boldsymbol{u}_i^h \cdot \boldsymbol{\mu}\, ds + \int_{\Gamma^{\mathrm{C}}_{\mathrm{N}}} (\boldsymbol{u}_i^h \cdot \boldsymbol{n})(\boldsymbol{\mu} \cdot \boldsymbol{n})\, ds \tag{6.9}$$

$$\boldsymbol{S} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}_{\mathrm{S}}} \frac{1}{\gamma} \boldsymbol{\lambda}^p \cdot \boldsymbol{\mu}\, ds + \int_{\Gamma^{\mathrm{C}}_{\mathrm{N}}} \frac{1}{\gamma} (\boldsymbol{\lambda}^p \cdot \boldsymbol{n})(\boldsymbol{\mu} \cdot \boldsymbol{n})\, ds \tag{6.10}$$

$$\boldsymbol{S}_i \longleftrightarrow \int_{\Gamma^{\mathrm{C}}_{\mathrm{S}}} \frac{1}{\gamma} \boldsymbol{\lambda}^p \cdot (\boldsymbol{\sigma}(\boldsymbol{v}_i) \cdot \boldsymbol{n})\, ds + \int_{\Gamma^{\mathrm{C}}_{\mathrm{N}}} \frac{1}{\gamma} (\boldsymbol{\lambda}^p \cdot \boldsymbol{n}) \sigma_n(\boldsymbol{v}_i)\, ds \tag{6.11}$$

$$\boldsymbol{S}_i^{\mathrm{T}} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}_{\mathrm{S}}} \frac{1}{\gamma} (\boldsymbol{\sigma}(\boldsymbol{u}_i^h) \cdot \boldsymbol{n}) \cdot \boldsymbol{\mu}\, ds + \int_{\Gamma^{\mathrm{C}}_{\mathrm{N}}} \frac{1}{\gamma} \sigma_n(\boldsymbol{u}_i^h)(\boldsymbol{\mu} \cdot \boldsymbol{n})\, ds \tag{6.12}$$

$$\boldsymbol{S}_{ij} \longleftrightarrow \int_{\Gamma^{\mathrm{C}}_{\mathrm{S}}} \frac{1}{\gamma} (\boldsymbol{\sigma}(\boldsymbol{u}_i^h) \cdot \boldsymbol{n}) \cdot (\boldsymbol{\sigma}(\boldsymbol{v}_j) \cdot \boldsymbol{n})\, ds. \tag{6.13}$$

The actual stick-slip region must be checked in the solution process in such a way that new integration limits $\Gamma^{\mathrm{C}}_{\mathrm{S}}$ (and $\Gamma^{\mathrm{C}}_{\mathrm{N}}$) are set up in each iteration step in case the magnitude of the tangential traction in the stick region exceeds $\mu\sigma_n$.

7. Numerical examples

7.1. **Hertz contact - no friction - global and local multipliers.** In this section we solve the contact problem of two bodies, a cylinder and a plane, that are coming into frictionless contact due to a compressive force $P = 100$ per unit length, see Figure 1. The width of the contact zone b and the distribution of the normal traction σ_n due to P are given by the Hertz theory [1]

$$\sigma_n(x) = \sigma_n^{max}\sqrt{1-\left(\frac{x}{b}\right)^2}, \tag{7.1}$$

where

$$b = 1.52\sqrt{\frac{PR}{E}}, \tag{7.2}$$

and

$$\sigma_n^{max} = 0.418\sqrt{\frac{PE}{R}}, \tag{7.3}$$

where x is measured from the in-plane symmetry line. R is the radius of the cylinder and E is the constant elastic modulus of both bodies where $R = 1$ and $E = 7000$. Equations (7.2) and (7.3) are valid in the case $\nu = 0.3$ for both bodies.

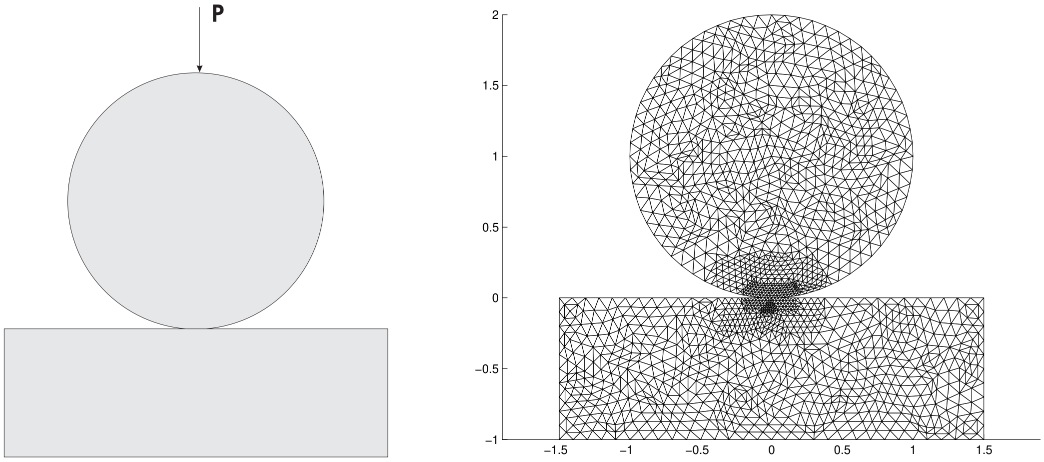

FIGURE 1. Hertz contact - Cylinder and a plane.

As to the support of the multiplier space, we let the global polynomials be defined on the surface of the plane within the zone $-a \le x \le a$ with $a = 0.2$. We have, as in [10], used global Legendre polynomials to avoid ill-conditioning of the polynomial approximation. Using global polynomials, the number of active multipliers is determined in advance and we do not change the coefficient matrix within the iteration. For the local multipliers we use piecewise constant multipliers defined on the surface of the plane.

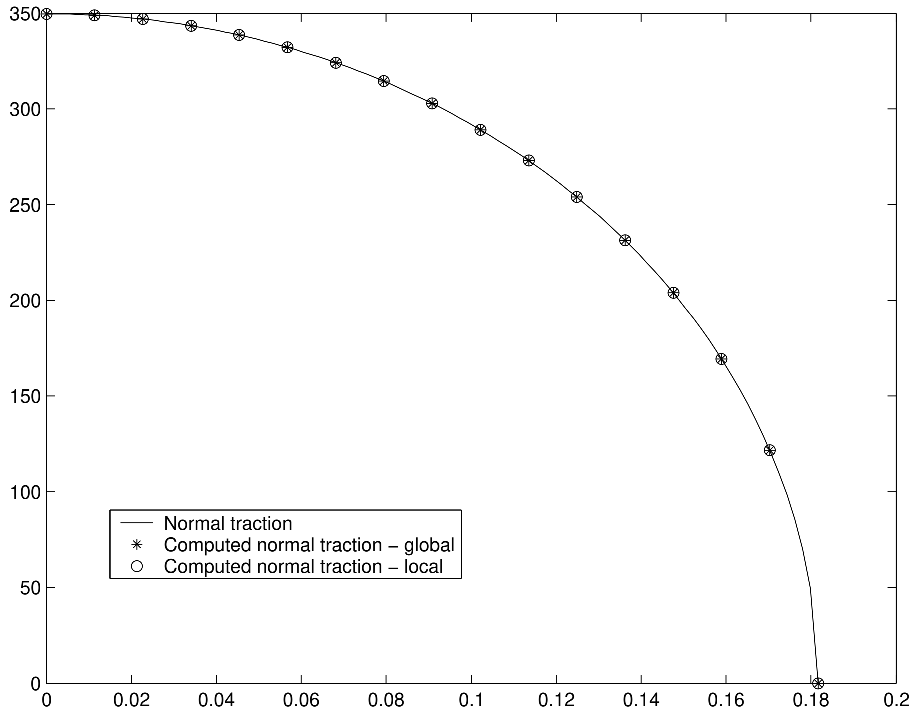

FIGURE 2. Normal traction distribution

7.2. **The Cattaneo-Mindlin problem - partial slip - local multipliers.** The method of solution to the problem of partial slip was first introduced by Cattaneo [2] and later also considered by Mindlin in [3]. Referring to Figure 3, two half cylinders are coming into contact by a compressive normal force $P = \int p(x)\, ds$ per unit length, to which a tangential force $Q = \int q(x)\, ds$ is subsequently applied. The width of the contact zone b and the pressure distribution due to P are given by the Hertz theory [1]

$$b = \frac{P(k_1 + k_2)R_1 R_2}{R1 + R2} \tag{7.4}$$

$$\sigma_n(x) = \sigma_n^{max}\sqrt{1 - \left(\frac{x}{b}\right)^2} \tag{7.5}$$

where $R1$ and R_2 are the radii of the cylinders and k_1 and k_2 are material parameters defined by

$$k_i = \frac{1 - \nu_i^2}{\pi E_i}. \tag{7.6}$$

and

$$\sigma_n^{max} = \sqrt{\frac{P(R_1 + R_2)}{\pi^2(k_1 - k_2)R_1 R_2}}. \tag{7.7}$$

Both cylinders are of isotropic linear elastic material with modulus $E = 7000$, poisons ratio $v = 0.3$ and the Coulomb friction coefficient is $\mu = 0.2$. The radii of both cylinders are $R = 1$. As to the boundary conditions, the upper cylinder is subjected to a constant traction vector $\boldsymbol{t}$ with components $\boldsymbol{t} = (p, q) = (50, 5)$.

Keeping p constant, and increasing q from zero, micro-slip evolves starting from the edges of the contact zone. The contact zone is thus divided into a stick part $-c \leq x \leq c$ and two areas of slip $c \leq |x| \leq b$. The size of the stick region is given by

$$c = b\left(1 - \frac{Q}{\mu P}\right)^{1/2}. \tag{7.8}$$

Within the contact zone, we have the following expressions for the tangential traction

$$\sigma_t(x) = \frac{\mu\sigma_n^{max}}{b}\left(\sqrt{b^2 - x^2} - \sqrt{c^2 - x^2}\right) \tag{7.9}$$

which is everywhere below the maximum tangential traction $\mu\sigma_n$. In Figure 4 the numerical solution is plotted against the analytical expressions given by eq.(7.5) and (7.9). The maximal tangential traction force $\sigma_t(x) = \mu\sigma_n(x)$ is also plotted in the figure.

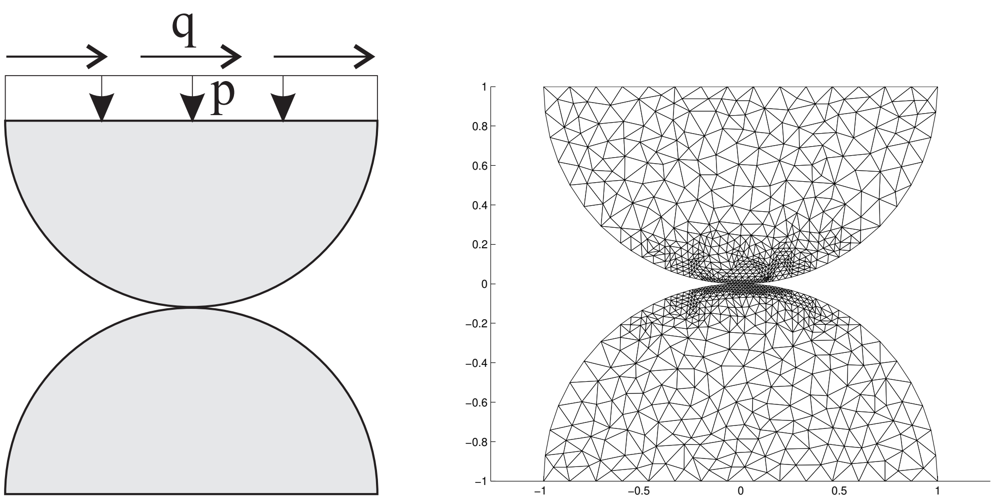

FIGURE 3. Contact between two cylinders

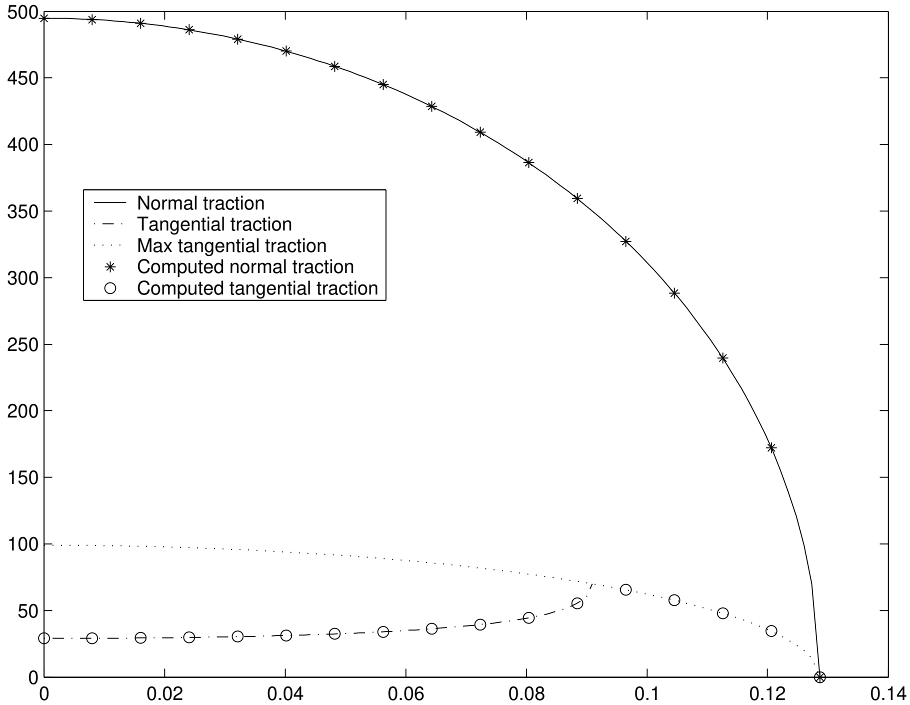

FIGURE 4. Contact between two cylinders

7.3. **Conclusions and further development.** In this paper, a new stabilized Lagrange multiplier method for elastic contact with Coulomb friction was presented. In the non-frictional case, the method has been proved to be stable independently of the discretization of the bodies that are coming into contact. Finally two examples that illustrates the performance of the method was presented. Further development would be to consider large deformations with inelastic material behavior and to incorporate an adaptive algorithm.

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