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IRREVERSIBILITY IN REVERSIBLE SYSTEMS II: THE INCOMPRESSIBLE EULER EQUATIONS

JOHAN HOFFMAN AND CLAES JOHNSON

Abstract. This is the second part of a series, where we present a new approach to resolving the classical paradox of irreversibility in reversible Hamiltonian systems. We base our solution on finite precision computation in the form of General Galerkin G2, instead of statistical mechanics. In the present Part II we consider as Hamiltonian model the Euler equations for an inviscid incompressible fluid. We show that the irreversibility arsises because G2 reacts by introducing a dissipative weighted least squares control of the residual if the Euler equations lack solutions with pointwise vanishing residual, which is the general case because of the appearance of turbulence.

1. Introduction

> There are great physicists who have not understood it.
> (Einstein about Boltzmann's statistical mechanics)

This is the second part of a series, where we present a new approach to resolving the classical paradox of irreversibility in reversible Hamiltonian mechanics based on Newton's Second Law. We base our solution on finite precision computation instead of statistical mechanics, which is the standard approach. We thus stay within a deterministic Hamiltonian framework and only add a restriction of finite precision computation, and we do not use any form of statistics. A World governed by Hamiltonian mechanics combined with finite precision computation, follows the laws of mechanics as far as possible taking the finite precision into account, but is not a game of roulette as in statistical mechanics. The difference of scientific paradigm is fundamental.

In Part I of the series we chose as a basic example of Hamiltonian mechanics the Euler equations for an inviscid compressible perfect gas with focus on model problems in one space dimension. We discussed the appearance of shocks in compressible inviscid flow as an example of an irreversible phenomen arising in a formally reversible system. In the present Part II we consider the Euler equations for incompressible inviscid flow in three dimensions, and show that in this case turbulence represents the irreversible phenomenon. In Part III

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we extend to compressible flow with a combination of shocks and turbulence as irreversible phenomena. We continue in Part IV with a study of the kinetic theory of gases, and we hope to ultimately approach also quantuum mechanics using the same computational deterministic point of view, again avoiding the conventional statistical interpretation.

We refer the reader to Part I for a broad introduction to the basic ideas using different perspectives of physics/mechanics, mathematics and computation. Here we give a compressed version of this exposition, and then proceed to the incompressible Euler equations.

The origins of irreversibility in reversible systems is a main unsolved mystery of mechanics and physics. A Hamiltonian system is reversible in time and does not have a preferred (forward) direction of time: From a given configuration both the future and past are equally well determined. The reversibility follows from the invariance of a Hamiltonian system under a change of sign of time and velocity. It follows in particular that letting a Hamiltonian system evolve in time from an initial configuration to a final configuration and there reversing the velocity and changing the direction of time, will bring the system back to the initial configuration. As a result, one may in Hamiltonian mechanics construct a perpetuum mobile of the first kind, which is a machine that will run forever without consuming any energy. Both celestial mechanics and quantum mechanics are Hamiltonian and the motion of the planets in our Solar system as well as the electrons in an atom represent reversible perpetuum mobile of the first kind.

On the other hand, in the real World there is a preferred direction of time and we are all familiar with irreversible processess in which initial configurations cannot be recovered, and the impossibility of constructing a perpetuum mobile of the first kind, as well as of the second kind supposed to reversibly convert energy back and forth from heat to mechanical work without consuming any net energy. The irreversibility is expressed in the Second Law of Thermodynamics, which states that in an isolated system a certain scalar quantity, named *entropy*, cannot decrease with time. As a consequence, an isolated system becomes irreversible if its entropy increases, since time reversal would correspond to decreasing entropy, which is impossible. In a Hamiltonian system the entropy is equal to minus the total energy being the sum of kinetic and potential energy, and energy conservation reflects reversibilty and entropy constancy. The observation that a perpetuum mobile of the second kind seems impossible, because converting mechanical energy into heat does not seem to be fully reversible, indicates the existence of real processes which are irreversible and thus not Hamiltonian. Dropping a stone to the ground will convert its potential energy into heat making the stone warmer, but the reverse process of the stone lifting itself by getting colder, is impossible. The question is why?

So if now the World ultimately is governed by reversible Hamiltonian (quantuum) mechanics, the scientific challenge thus becomes to explain how irreversibility may arise in systems based on reversible Hamiltonian mechanics. In the late 19th century when the existence of an Aether filling empty space was still contemplated, the irreversibilty was suggested to possibly result from some small viscosity of the Aether, but since no one could ever detect any Aether, this belief faded. Similarly, the idea of putting in just a tiny bit of friction (coming from somewhere) to explain irreversibility, is not convincing, since then the planets and electrons would be constantly retarding a little bit, but they don't seem to do that. And if there would be some friction in the system, the challenge would be to explain how friction can arise in a system governed by Hamiltonian reversible mechanics without friction. Thus the irreversibility paradox can be phrazed: How can there be friction in a system without friction?

The traditional way to resolve the paradox has been statistical mechanics, which is an expansion of Hamiltonian mechanics using concepts from statistics and probability. This expansion has a high scientific cost, since so many new (difficult) questions arise from the use of statistics. Accordingly statistical mechanics has been questioned by many famous scientists including Einstein, and still is.

Altogether, as far as we can understand, the true origins of irreversibility in reversible systems has not been given a scientifically convincing explanation. The literature is vast with contributions from mathematicians, physicists, chemists, engineers, philosophers, linguists, authors of science fiction and the general public.

2. Finite Precision Computation

We now focus on the new mode of explanation based on finite precision computation, which we advocate. The finite precision computation appears in two forms: First, it necessarily appears in digital solution of Hamiltonian equations using computers, which is the objective of our study. Secondly, it probably appears also in Natures evolution in time from one state to the next in some form of analog computation.

The solution of the paradox of irreversibility in reversible system based on finite precision computation, is not trivial in the sense that it may be blamed simply on something like round-off errors in digital computing or the inevitable approximations in solving differential equations numerically. This would be similar to explaining irreversibility as an effect of a slightly viscous Aether, a mode of explanation we have already rejected.

The solution of the paradox is much deeper and more fundamental and directly couples to our recent work on computational turbulence exposed in [4]. In short, the secret we uncover is the following: We consider a set of Hamiltonian equations describing the evolution in space/time of a certain system in Nature. We seek to solve the equations computationally using a numerical method implemented on a computer. Doing so we meet two different situations: In the first case, which is the simple standard case without surprise, the Hamiltonian equations have pointwise solutions which are computable, and if so we simply compute these solutions and find them to be reversible. A pointwise solution has a residual which is pointwise zero, obtained by inserting the solution in the equation, and we can compute approximate solutions with residuals being small pointwise. Such computed solutions are approximately reversible by the reversible nature of the equations they are approximately solving pointwise.

In the second case, which contains the secret, the Hamiltonian equations do not admit pointwise solutions, which means that there simply are no (stable) solutions with a residual being zero pointwise. This reflects the appearance of small scale phenomena such as turbulence and/or shocks in the case of inviscid fluid mechanics. In this second case the computational method cannot produce an approximate solution with small pointwise residual, and the computational method we are using reacts by producing an approximate solution for which the residual is small in a weak average sense combined with a certain weighted least squares control of the residual, which turns out to be possible to achieve. We refer to the numerical method with this property as General Galerkin or G2. In the case the Hamiltonian equations do not admit pointwise solutions, which may correspond to the appearence of turbulence and/or shocks, G2 thus produces an approximate solution with the residual being small in a weak sense and with a certain weighted least squares control of the size of the pointwise residual, while the pointwise residual itself is not small.

We shall see that this is about the best that can be done in the situation when the Hamiltonian equations do not admit pointwise solutions, but we shall also see that it is good enough if we as quantities of interest or output quantities choose certain mean values of the solution, rather than point values. In the case the Hamiltonian equations do not admit pointwise solutions, corresponding to turbulence/shocks, we can thus nevertheless by G2 compute certain mean value outputs accurately. From a physical point of view, we may say that even though the Hamiltonian equations cannot be satisfied pointwise, they can be satisfied in an average sense with the pointwise residual not being too large, and that may be enough for the system to evolve. The pointwise violation but average satisfaction of the Hamiltonian laws in this sense, corresponds to a physical system in pointwise non-equilibrium, but in average local equilibrium with some control of the pointwise non-equilibrium. In such a physical system the laws of physics serve as goals, which cannot be satisfied pointwise, and the search of satisfaction in a suitably approximate sense is what drives the evolution of the system. It is like the Law in our society, which is never followed pointwise by all citizens, only in some average sense, but yet has an important role to secure that society does not fall apart.

Now, the catch is that the weighted least squares control of the residual in G2 adds a dissipative term in an energy balance, which effectively makes the system irreversible. This is like a fine or cost arising from not following the Law pointwise. It is thus the appearance of turbulent/shock small scales and the resulting impossibility of computing solutions with pointwise small residuals, which necessarily introduces the irreversibility. By necessity, a fine has to represent a positive cost; if we would get paid by breaking the Law, society would quickly collapse. Or if there would be a negative cost (gain) in changing currency, the monetary system would explode.

Facing the impossibility of pointwise solution, the system thus reacts by producing an approximate solution in which some of the energy is lost in a dissipative least squares term implying irreversibility. Moreover, the size of the dissipation and the energy loss do not decrease with increasing precision: In turbulence the dissipation always occurs on the finest scales available, but the total amount of the turbulent dissipation (turning into heat), stays (approximately) constant under scale refinement. A shock in compressible flow has a similar nature. Mean value outputs thus may show an independence of the scale of resolution in the computation, while pointwise solution is impossible even if the computational scale is refined indefinitely. The more you refine, the more scales you find and there is no end to this process.

The basic idea is thus that in certain Hamiltonian processes necessarily small scale features in the form of turbulence/shocks appear, and when faced with these small unresolvable scales, which physically generate heat, the system reacts by introducing a dissipative least squares control of the residual, which implies irreversibility. Thus, in turbulence/shocks, large scale mechanical energy may be turned into small scale motion, corresponding to generation of heat, and this process is irreversible since the details of the small scales cannot be kept and thus cannot be recovered.

The key here is to realize that the dissipative damping (i) is necessary, (ii) is substantial, and (iii) is not a numerical artifact which can be diminished by increasing the precision. The key new fact behind (i)-(iii) is the non-existence of solutions to the Hamiltonian equations!

The appearance of turbulence/shocks in inviscid compressible flow is an example of an irreversible process satisfying (i)-(iii), where inevitably and irreversibly energy is turned into heat. As is well known, a shock solution is a not pointwise solution to the Euler equations. As we will show below, neither does turbulence correspond to a pointwise solution.

In G2 the irreversibility arises from the presence of the least squares control of the residual, which corresponds to a loss of the kinetic/potential energy which cannot be recovered in G2; reversing time and velocities at final time in G2 and computing backwards in time will bring in a new least squares term only adding to the losses already made in the forward computation. This reflects the difficulty of getting a refund of an already paid fine.

3. The Second Law of Thermodynamics

We may summarize our results as proving that the Second Law of Thermodynamics is a consequence of the First Law of Thermodynamics (which expresses conservation of energy) combined with finite precision computation. We may thus propose a new foundation of thermodynamics based on deterministic mechanics expressed by the First Law combined with finite precision computation, as opposed to a usual foundation with the Second Law as an additional postulate. We will return to this task in more detail.

Finite precision computation of course appears in digital solution of the differential equations of deterministic mechanics, but it necessarily also has to appear in some form in the analog computation performed in the physics of the real World. We may analyze the consequences of finite precision computation of digital solution, and then seek to find analogs in physics.

This brings us back to a deterministic World as a giant Clock in the spirit of Laplace, but our Clock has finite precision and that changes the game. In particular, it takes us out of the classical paradox of the existence of free will in a deterministic World. With finite precision computation, the future is no longer fully determined by the present, and there is room for something like a free will. And there are necessarily irreversible processes.

4. The Euler Equations for Fluid Flow

The Euler equations for incompressible inviscid flow may be viewed to model a very large collection of “fluid particles” following Newton’s Second Law subject to a pressure force maintaining incompressibility.

The incompressible Euler equations represent a formally reversible system, which as we will see in general lacks pointwise solutions. This is because the laminar pointwise solutions, which do exist, turn out to be unstable without physical realization, and because the turbulent solutions, which do appear, are not pointwise solutions but only weak approximate solutions. Thus, both computation and Nature will have to go for suitable approximate solution of the Euler equations. Computation will then rely on G2, with presumably Nature resorting to something similar, which inevitable (because of the least squares residual control in G2) will introduce a dissipative effect implying irreversibility.

We will thus study a situation, where the equations we want to solve do not have exact pointwise solutions (or if they have, then they are unstable), while the turbulent solutions which do exist in fact only are approximate weak solutions and not pointwise solutions, and moreover these approximate solutions necessarily have a dissipative character resulting in irreversibility. The paradox of irreversibility in a formally reversible Hamiltonian system is thus a consequence of the non-existence of stable laminar pointwise (strong) solutions to the Euler equations, which would have been reversible if they had only existed, and the dissipative nature of the turbulent approximate weak solutions, which do exist computationally and for which mean value outputs can be accurately computed.

We note that the non-existence of (stable) exact solutions changes the way mathematics for the Euler equations can be presented: With non-existent exact solutions, the attention has to move to existing approximate solutions, and thus the computational aspect takes a prime position before analytical mathematics.

The non-existence of pointwise solutions to the Euler equations, which may be viewed as a failure of mathematics, in fact may be turned around into an advantage from a computational point of view: If there were an exact solution, one could always ask for more precision in computing this solution requiring finer resolution and higher computational cost, but if there is no exact solution, then we could be relieved from this demand beyond a certain point. A key feature in this situation is that the absolute size of the fine scales no longer are important, and this could save computational work. In turbulence this means that mean value outputs may be computed on meshes which do not resolve the turbulent vortices to their actual physical scale.

In order for a Hamiltonian system to develop turbulence, it has to be rich enough in degrees of freedom. In particular, the incompressible or compressible Euler equations in less than three space dimensions are not rich enough, even if the mesh is very fine. On the other hand, turbulence invariably develops in three dimensions once the mesh is fine enough. Our experience with turbulent solutions of the incompresible Navier-Stokes equations indicates that a mesh with 100 000 mesh points in space may suffice in simple geometries, while in more complex geometries millions, but not billions, of mesh points may be needed.

5. Imperfect Nature and Mathematics?

How are we to handle the fact that the Euler equations do not have pointwise solutions in general? Does this express an imperfection of mathematics? And what is the consequence in physics? Is Nature simply unable to satisfy the basic laws laid down in the form of e.g. Newton's Second Law? Does this mean that also Nature is imperfect? And if now both mathematics and Nature indeed are imperfect, what is the degree of imperfection and how does it show up?

We may make a parallel with the squareroot of two $\sqrt{2}$, which is the length of the diagonal in a square with side length 1. We know that the Pythagoreans discovered that $\sqrt{2}$ is not a rational number. This knowledge had to be kept secret, since it indicated an imperfection in the creation by God formed as relations between natural numbers according the basic belief of the Pythagoreans. Eventually this unsovable conflict ruined their philosophical school and gave room for the Euclidean school based on geometry instead of natural numbers. Civilization did not recover until Descartes resurrected numbers and gave geometry an algebraic form, which opened for Calculus and the scientific revolution.

But how is the Pythagorean paradox of non-existence of $\sqrt{2}$ as a rational number handled today? Well, we know that the accepted mathematical solution since Cantor and Dedekind is to extend the rational numbers to the real numbers, some of which like $\sqrt{2}$ are called irrational, and which can only be described approximately using rational numbers. We may say that this solution in fact is a kind of non-solution, since it acknowledges the fact that the equation $x^2 = 2$ cannot be solved exacly using rational numbers, and since the existence of irrational numbers (as infinite decimal expansions of Cauchy sequences of rational numbers) has a different nature than the existence of natural numbers or rational numbers. The non-existence is thus handled by expanding the solution concept until existence can be assured.

We handle the non-existence of pointwise solutions to the Euler equations similarly, that is, by extending the solution concept to an approximate solution in a weak sense combined with some control of pointwise residuals. Doing so we necessarily introduce a dissipation causing irreversibility. In this case, the non-existence of solutions thus has a cost: irreversibility. In the perfect World, pointwise solutions would exist, but this World cannot be constructed neither mathematically nor physically, and in a constructible World necessarily there will exist irreversible phenomena as a consequence of the non-existence of pointwise solutions. The non-existence of pointwise solutions reflects the development of complex solutions with small scales, and thus the non-existence also relects a complexity of the constructible World. The perfect World would lack this complexity, so in addition to being non-existent it would also probably be pretty non-interesting. The World we live in thus does not seem to be perfect, but it surely is complex and interesting.

What is the reason that the resolution of the paradox we are proposing has not been presented before, if it indeed uncovers the mystery? We believe it can be explained by the Ideal Worlds that both mathematicians and physicists assume as basis of their science. In the Ideal World of mathematics, exact solutions to differential equations exist as well as infinite sets, not just approximate solutions and finite sets, and the World of physics is supposed to follow laws of physics exactly, not just approximately, unless a resort to statistics is made (which is a very strong medication with severe side effects). It thus appears that an imperfect World of mathematics or physics, where equations cannot be solved exactly or laws of physics cannot be exactly satisfied, classically is unthinkable at least as a deterministic World, and thus has recieved little attention by mathematicians and physicists with little background in computational mathematics. Yet, such an imperfect World seems to be a reality in both mathematics and physics, and thus should be studied.

6. A New Paradigm?

From philosophical point of view, we may say that the traditional paradigm of both mathematics and physics is Platonistic in the sense that it assumes the existence of an Ideal World, where equations/laws are satisfied exactly. We may say that this is an Ideal World of infinities because exact satisfaction of e.g. the equation $x^2 = 2$ requires infinitely many decimals. This is the mathematical Ideal World of Cantor, which represents a formalist/logicist school. In strong opposition to this school of infinities, is the constructivist school, which only deals with mathematicial objects that can be constructed in a finite number of steps. In the constructivists Constructible World, the set of natural numbers does not exist as a completed mathematical object as in Cantors Ideal World, but only as a never-ending project where always a next natural number can be constructed if needed, which follows the suggestions of e.g. Aristotle and Gauss. The Constructible World is finitary and thus inherently computational, while Cantors Ideal World is non-finitary and non-computational. In the educational project [8] and the pamphlett [9], we compare the two schools, and give our vote to the Constructible World, which today can be explored using the computer, and we question the existence an Ideal World as always a scientifically meaningful concept.

7. A $1 Million Prize Problem

One of the seven Clay Institute Millennium $1 Million Prize Problems asks for a proof of existence of a pointwise solution to the Navier-Stokes equations for incompressible fluid flow, a formulation which fits into an Ideal World paradigm. In [9] we claim that the formulation of the Prize Problem is unfortunate and should be reformulated in constructive terms, since in general pointwise solutions do not exist, while turbulent approximate solutions do. We note that the Clay Institute does not react to this criticism, which could be viewed as an analog of the Pythagoreans denial of non-existene of an exact solution to the equation $x^2 = 2$. What will the consequence be when the secret of non-existence of pointwise solutions to the Navier-Stokes equations is broken?

8. Physics vs Computation

We have noted that the mechanism making G2 irreversible when applied to a sufficiently complex formally reversible Hamiltonian system, is the least squares control of the pointwise residual introducing a dissipative effect when pointwise solutions do not exist. It is natural to believe that Nature resorts to something similar, but the more precise physics of this effect is of course up to debate and study. In general, one may view the physics/mechanics of a system of interacting particles as some kind of analog computation, where during each little time step the particles exchange data concerning (relative) positions and forces determining accelerations and then update velocities, positions and forces for the next time step. But the more exact nature of the exchange process is largely unknown, and it is conceivable that a careful study of computational models may open doors to understanding, as suggested by the famous computer scientist Dijkstra: *Originallly I viewed it as the function of the abstract machine to provide a truthful picture of the physical reality. Later, however, I learned to consider the abstract machine as the "true" one, because that is the only one we can "think"; it is the physical machine's purpose to supply a "working model", a (hopefully) sufficiently accurate physical simulation of the true, abstract machine.*

9. Outline

We start by recalling the Navier-Stokes equations for viscous incompressible flow and discuss basic aspects of turbulent solutions, and then proceed to the Euler equations obtained by setting the zero viscosity to zero and changing from no-slip to slip boundary conditions.

We also present a solution to d'Alemberts paradox stating that the drag of a bluff body subject to inviscid incompressible flow is zero. The resolution builds on the fact that the pointwise solution to the Euler equations used by d'Alembert in his computation of zero drag, is unstable and thus cannot be realized and thus in fact is non-existent. Obviously, this makes d'Alembert's computation invalid. Computing instead approximate solutions of the Euler equations using G2, we find a non-zero drag which is close to the drag of a slightly viscosus fluid.

10. The Incompressible Navier–Stokes Equations

The Navier–Stokes equations for an incompressible Newtonian fluid of constant unit density and constant kinematic viscosity $\nu > 0$ enclosed in a volume Ω in R^3 with boundary Γ over a time interval $I = (0, T]$, read as follows:

$$\begin{aligned} \dot{u} + (u \cdot \nabla)u - \nu\Delta u + \nabla p &= f && \text{in } \Omega \times I, \\ \nabla \cdot u &= 0 && \text{in } \Omega \times I, \\ u &= 0 && \text{on } \Gamma \times I, \\ u(\cdot, 0) &= u^0 && \text{in } \Omega, \end{aligned} \tag{10.1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the *velocity* and $p = p(x,t)$ the *pressure* of the fluid at $(x,t) = (x_1, x_2, x_3, t) \in \Omega \times I$ with u_i the velocity in the coordinate direction x_i, and the dot indicates differentiation with respect to time. Further, $f = (f_1, f_2, f_3)$ is a given volume force acting on the fluid, and $u^0 = u^0(x)$ is a given initial velocity. We here assume for definiteness homogeneous Dirichlet boundary conditions for the velocity u. We further assume that ν is small, and that a typical velocity u and length scale is of unit size, so that the Reynolds number $\sim \nu^{-1}$ is large. The first equation in (10.1) expresses Newton's Second Law of conservation of momentum in vector form, and the second equation expresses conservation of mass in the form of incompressibility.

For an incompressible flow described by a solution $\hat{u} = (u, p)$ of the Navier–Stokes equations, the Second Law of Thermodynamics follows from the following basic energy balance obtained by multiplying the momentum equation by u and integrating (assuming $f = 0$ for simplicity):

$$\frac{1}{2}\|u(T)\|^2 + D_\nu(u, T) = \frac{1}{2}\|u(0)\|^2, \tag{10.2}$$

where $u(t) = u(\cdot, t)$, $\|\cdot\|$ is the $L_2(\Omega)^3$-norm, $\frac{1}{2}\|u(t)\|^2$ is the kinetic energy at time t, and

$$D_\nu(u, T) = \sum_{i=1}^{3} \int_0^T \nu \|\nabla u_i\|^2 dt \tag{10.3}$$

represents the quantity which physically is dissipated into heat.

The entropy for a Navier-Stokes solution $\hat{u}$ at time T is simply $-\frac{1}{2}\|u(T)\|^2$ and the increase of entropy from $-\frac{1}{2}\|u(0)\|^2$ is equal to the dissipative term $D_\nu(u, T)$. The characteristic feature of a turbulent solution $\hat{u}$ is that $D_\nu(u, T)$ is not small, signifying a considerable entropy production or generation of heat corresponding to small scale velocity fluctuations with $|\nabla u_i|$ large. Increasing entropy thus corresponds to turbulent dissipation into heat in the form of small scale velocity fluctuations, which are irreversible because the backward heat equation is unstable for highly oscillatory data. On the other hand, in a laminar flow $D_\nu(u, T)$ will be very small and the entropy will be very slowly increasing, but it will increase since $\nu > 0$.

We conclude that in a World governed by the Navier–Stokes equations with viscosity $\nu > 0$, the entropy would be increasing more or less, and in this World we would not find planetary systems and atoms with constant entropy. Since we undoubtely observe such phenomena, we would be led to set $\nu = 0$, which leads us to the next section.

11. The Incompressible Euler Equations

The Euler equations for an incompressible flow are obtained setting the viscosity $\nu = 0$ in the Navier–Stokes equations (10.1) and changing the Dirichlet boundary conditions to get:

$$\begin{aligned}
\dot{u} + (u \cdot \nabla)u + \nabla p &= f && \text{in } \Omega \times I,\\
\nabla \cdot u &= 0 && \text{in } \Omega \times I,\\
u \cdot n &= 0 && \text{on } \Gamma \times I,\\
u(\cdot, 0) &= u^0 && \text{in } \Omega,
\end{aligned} \tag{11.1}$$

where n is the outward unit normal to Γ. We note that in the Euler equations only the normal velocity $u \cdot n$ is prescribed to be zero on the boundary, while the tangential velocity is free. This is also referred to as a slip boundary condition, as compared to the no-slip Dirichlet boundary condition $u = 0$ in the Navier–Stokes equations.

The Euler equations are formally reversible: Changing the sign of time t and the velocity u, obviously leave the equations unchanged. In particular, if $\hat{u}$ is a solution to the Euler equations with initial velocity u^0 at time $t = 0$ and final value $\hat{u}(T)$ at time $t = T$, then the function $\hat{v}(t) = (-u(T-t), p(T-t))$ satisfies Eulers equations for $t \in (0, T]$ with initial data $v(0) = -u(T)$ and final velocity $v(T) = -u(0)$. Thus reversing the velocities $u(T)$ and letting time pass backwards would bring back the velocities to $-u(0)$ from $-u(T)$.

The basic energy estimate (10.2) with $\nu = 0$ states that the kinetic energy (negative entropy) $\frac{1}{2}\|u(t)\|^2$ stays constant if $f = 0$:

$$\frac{1}{2}\|u(T)\|^2 = \frac{1}{2}\|u(0)\|^2. \tag{11.2}$$

It follows that a system governed by the Euler equations allows the design of a *perpetuum mobile* by reversing the velocity at $t = 0$ and $t = T$, corresponding to a system bouncing back and forth for ever. We conclude that if the Euler equations admit a pointwise solution with pointwise zero residual, then that solution would be reversible and represent a perpetuum mobile.

However, as we will see, in general the Euler equations do not have pointwise solutions, so any conclusion made from an assumption of existence of a pointwise solutions is risky, including the energy conservation (11.2), as we will see. We use this insight below to resolve d'Alemberts paradox, by showing that the pointwise solution used by d'Alembert to compute zero drag, simply does not exist!

Non-existence of pointwise solutions of the Euler equations follows from the observation that solutions to the Navier-Stokes equations in general are turbulent if ν is small, and that it is unthinkable that these turbulent solutions could converge to a pointwise solution of the Euler equations as ν tends to zero. The reason is that as we let ν tend to zero, the corresponding Navier-Stokes solutions develop ever finer scales of turbulence which is incompatible with convergence to a pointwise solution of the Euler equations. If the Navier-Stokes solutions had stayed laminar as ν tends to zero, pointwise convergence would have been possible, but the Navier-Stokes solutions invariably become turbulent if ν is small, and thus convergence simply does not take place. We thus have clear evidence that in general pointwise solutions of the Euler equations are non-existent.

For the analytical mathematical struggle to come to grips with the Euler equations, we refer to [1] and references therein.

12. SOLUTION OF THE EULER EQUATIONS BY G2

If it now is impossible to solve the Euler equations exactly pointwise and if Nature faces the same difficulty in its own analog computation seeking to follow Newton's Second Law and maintain incompressibility, the question is what the alternative to a pointwise exact solution could be? In a computational approach we suggest to use G2, which is a combination of Galerkin's method and a weighted least squares method, as presented in [4].

G2 takes the general form: Find $\hat{U} = (U, P) \in V_h$ such that

$$(R(\hat{U}), \hat{v})_Q + (hR(\hat{U}), R(\hat{v}))_Q = 0 \quad \text{for all } \hat{v} \in V_h, \tag{12.1}$$

where V_h is a space of piecewise polynomials on a mesh in space/time with mesh size h, $(\cdot,\cdot)_Q$ is the $=L_2(Q)^4$-scalar product with $Q=\Omega\times I$, $R(\hat U)=(\dot U+(U\cdot\nabla)U+\nabla P-f,\nabla\cdot U)$ and $R(\hat v)=(\dot v+(U\cdot\nabla)v+\nabla q,\nabla\cdot v)$. It is here not necessary to go into details, which in particular contain jump terms arising from discontinuous approximation in time, or a modification of the test functions in case $\hat U$ is kept continuous in time.

The first term in (12.1) is the Galerkin term asking the residual $R(\hat U)$ to vanish in a weak sense, and the second term is the weighted least squares stabilization with weight equal to the mesh size h. Choosing $\hat v=\hat U$, we obtain the basic energy estimate, assuming $f=0$:

$$\frac{1}{2}\|U(T)\|^2+\|\sqrt{h}R(\hat U)\|_Q^2=\frac{1}{2}\|U(0)\|^2, \tag{12.2}$$

where $\|\cdot\|_Q$ is the $L_2(Q)^4$-norm. Obviously, the least squares term $\|\sqrt{h}R(\hat U)\|_Q^2$ corresponds to the the viscous term $D_\nu(u,T)$ in the energy estimate for the Navier–Stokes equations, and thus corresponds to the generation of heat, as we return to below. Accordingly, we shall see that in case of turbulence, the least squares term is not small, while it is for laminar solutions.

We now choose a certain (mean value) quantity of interest $M(\hat U)$ depending on the G2 solution $\hat U$ such as drag or lift, and seek to estimate the difference in output of two different G2 solutions $\hat U$ and $\hat W$ on different meshes. Using duality one can show as in [4] that

$$|M(\hat U)-M(\hat W)|\le S(\|hR(\hat U)\|_Q+\|hR(\hat W)\|_Q),$$

where h represents the larger of the two mesh sizes, and S represents a stability factor obtained solving a dual problem. We can thus estimate the difference in output between two different G2 solutions in terms of their residuals multiplied with a certain stability factor. It is important to notice the presence of the crucial factor h multiplying the residuals, which makes it possible for the difference in output to be small, although the residual is not small pointwise. The reflection in G2 of non-existence of a pointwise solution is that $\|R(\hat U)\|_Q$ is not small, while $\|hR(\hat U)\|_Q$ may be small. Typically, $\|R(U)\|_Q\sim h^{-1/2}$, reflecting that the least squares term $\|\sqrt{h}R(\hat U)\|_Q$ has a significant contribution in the energy balance (12.2).

It is thus the impossibility of solving the Euler equations pointwise, which forces G2 to introduce a dissipative weighted least squares term to come to grips with an impossible situation. It is not difficult to envision that Nature faces the same problem and resorts to a similar type of solution involving a weak satisfaction of the balance laws together with some control of the pointwise dis-satisfaction. What else could there be to do in a situation when pointwise satisfaction is impossible? Of course, the dissipative least squares term puts a bound on solution gradients and thus destroys very fine scales which corresponds to information loss and increase of entropy.

Notice that it is the combination of Galerkin and weighted least squares that produces a reasonable compromise in the difficult case when a pointwise solution is impossible. Only least squares will not work because the residual cannot be small in the $L_2(Q)$-norm, and from only the knowledge that the residual is large nothing can be concluded. Further, only Galerkin will not work either because the residual control is too weak to produce any sensible output. It is only the combination of Galerkin and weighted least squares that works. The evidence of success is the presence of the factor h in the expression $\|hR(\hat{U})\|$ and the fact that by (12.2), we have that $\|hR(\hat{U})\| \leq \sqrt{h}$ if $\|u^0\| = 1$. In a pure least squares method the factor h in front of $R(\hat{U})$ would be missing, and in pure Galerkin one may have $R(\hat{U}) \sim 1/h$ and thus $\|hR(\hat{U})\| \sim 1$. Thus neither extreme case can work in general.

13. d'Alemberts Paradox

d'Alemberts paradox states that a bluff body subject to inviscid flow has zero drag, which is at variance with observations of a substantial drag even if the viscosity is very small.

We first recall d'Alemberts computation of zero drag: Suppose there is a pointwise (laminar) solution (u, p) to the Euler equations of inviscid incompressible flow around a bluff body in a horisontal channel oriented in the x_1-direction. Integrating the momentum equation over the domain, we obtain by partial integration, considering the first component

$$0 = \int_{\Gamma_b} pn_1\, ds + \int_{\Gamma_{in}} (u \cdot nu_1 + pn_1)\, ds + \int_{\Gamma_{out}} (u \cdot nu_1 + pn_1)\, ds$$

where Γ_{in} and Γ_{out} denote the inflow and outflow boundaries of the channel, and Γ_b denotes the boundary of the immersed body. Assuming now that the velocity is equal on inflow and outflow, which is natural if the channel is long, by Bernoullis law the pressure will be as well, and thus the inflow and outflow terms will cancel and therefore the drag $\int_{\Gamma_b} pn_1\, ds$ will be zero.

Obviously, zero drag of a bluff body contradicts experience: All bluff bodies show substantial drag with the major contribution coming from the pressure distribution around the body with high pressure in front and low pressure in the back, and not from viscosity. In particular, we can attribute only a small part of the drag to viscosity and thus experience clearly indicates substantial drag for inviscid flow. But d'Alemberts computation shows zero drag.

We shall now see that the trouble with d'Alemberts computation of zero drag is that the pointwise laminar solution simply does not exist as a stable solution, which makes the computation meaningless. Instead a turbulent approximate solution develops and this solution has a substantial drag. If a Maxwell Demon was able to stabilize the laminar solution, zero drag would result, but such a device seems impossible to realize.

14. Drag of a Square Cylinder

As a basic example we consider the problem of computing the drag of a square cylinder of diameter $D = 0.1$ centered at $x = (0.5, 0.7, 0.2)$ and oriented in the x_3-direction in a channel of dimension $2.1 \times 1.4 \times 0.4$ oriented in the x_1-direction, subject to a uniform inflow velocity. We use slip boundary conditions both on the cylinder and the channel walls, and we use a locally refined tetrahedral mesh with 86 904 mesh points, shown in Fig. 1.

We first determine a potential irrotational solution, not by solving the Euler equations, but looking for a stationary velocity $u = (u_1, u_2, 0)$ given by $u_1 = \frac{\partial\varphi}{\partial x_1}$, $u_2 = \frac{\partial\varphi}{\partial x_2}$, where $\varphi = \varphi(x_1, x_2)$ solves Laplace equation $\Delta\phi = 0$ in the domain of the fluid restricted to the $x_3 = 0$ plane with $\varphi = 0$ at inflow, $\varphi = 1$ at outflow, and with homogeneous Neumann conditions on the channel walls and the cylinder. Such a velocity u is irrotational with $\nabla\times u = 0$, and satisfies the stationary Euler equations $(u\cdot\nabla)u = 0$ and $\nabla\cdot u = 0$ in the fluid domain, with $u \cdot n = 0$ on channel walls and the cylinder surface, and has approximately equal inflow and outflow velocities. Note that the velocity u is two-dimensional with $u_3 = 0$. The corresponding pressure p is constant. The stationary solution $\hat{u} = (u, p)$ clearly exists and is smooth off the corners of the cylinder, and thus $\hat{u}$ represents a laminar solution with pointwise zero residual. We note that $\hat{u}$ is symmetric in the x_1-direction modulo the non-symmetric position of the cylinder. In particular, the flow before and after the cylinder is symmetric. The drag of this solution is zero since the pressure is constant, without pressure drop over the cylinder.

In practice we compute φ by solving $\Delta\varphi = 0$ using piecewise linear finite elements in the three-dimensionl fluid volume, and then associate a corresponding piecewise linear velocity $U^0 = \nabla\varphi$ by interpolation of the piecewise constant $\nabla\varphi$ to the nodes in the mesh. This produces an approximate potential solution $\hat{U}^0$ with $R(\hat{U}^0)$ being small pointwise except close to the edges of the cylinder.

We compute an approximate solution $\hat{U} = (U, P)$ to the Euler equations with initial velocity and inflow data given by U^0 using G2 in the form cG(1)cG(1) with continuous linear trial functions in space-time. We find that the computed velocity $U(t)$ remains equal to U^0 only for a few time steps, then develops non-symmetry in x_1 while maintaining two-dimensionality after which it successively develops into a fully three-dimensional turbulent solution which is far from irrotational, see Fig. 8. This turbulent solution is similar to the turbulent solution of the Navier-Stokes equations with small viscosity and with no slip boundary conditions on the cylinder presented in [2, 3].

In Fig. 2-3 we plot the solution (U, P) for the first few time steps, using a very small time step of size 0.1 times the smallest element diameter in the mesh. We find that the instability of the the initial symmetric solution $U(0) = U^0$ is first expressed in a fluctuating pressure until a high pressure in front of the cylinder is established, which initializes the development of a non-symetric velocity eventually going turbulent.

We show results in Fig. 4-8 starting with zero initial velocity, using now time steps of the same size as the finest element diameter in the mesh. We find again the potential solution during the first few time steps with the same development into a turbulent solution. In Fig. 1 we plot the time series of the normalized drag force, corresponding to the drag coefficient c_D, for the Euler solution and a Navier-Stokes solution with $\nu^{-1} = 22\ 000$ computed on the same mesh, where we find that the drag is very similar, of the order 2.0-2.5.

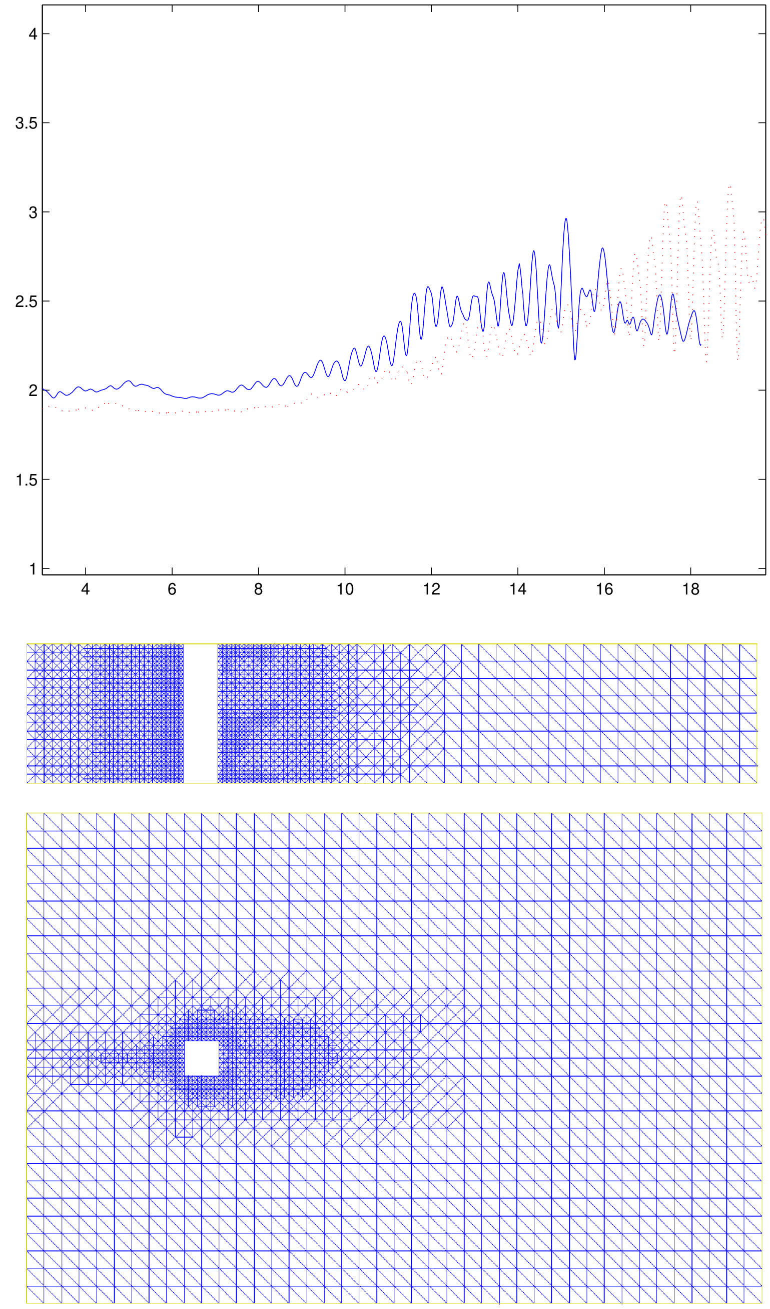

Figure 1. Normalized drag force for solutions to the Euler equations ('-') and the Navier-Stokes equations with $\nu^{-1} = 22\ 000$ (':'), and the corresponding computational mesh in the x_1x_2-plane (upper) and the x_1x_3-plane (lower).

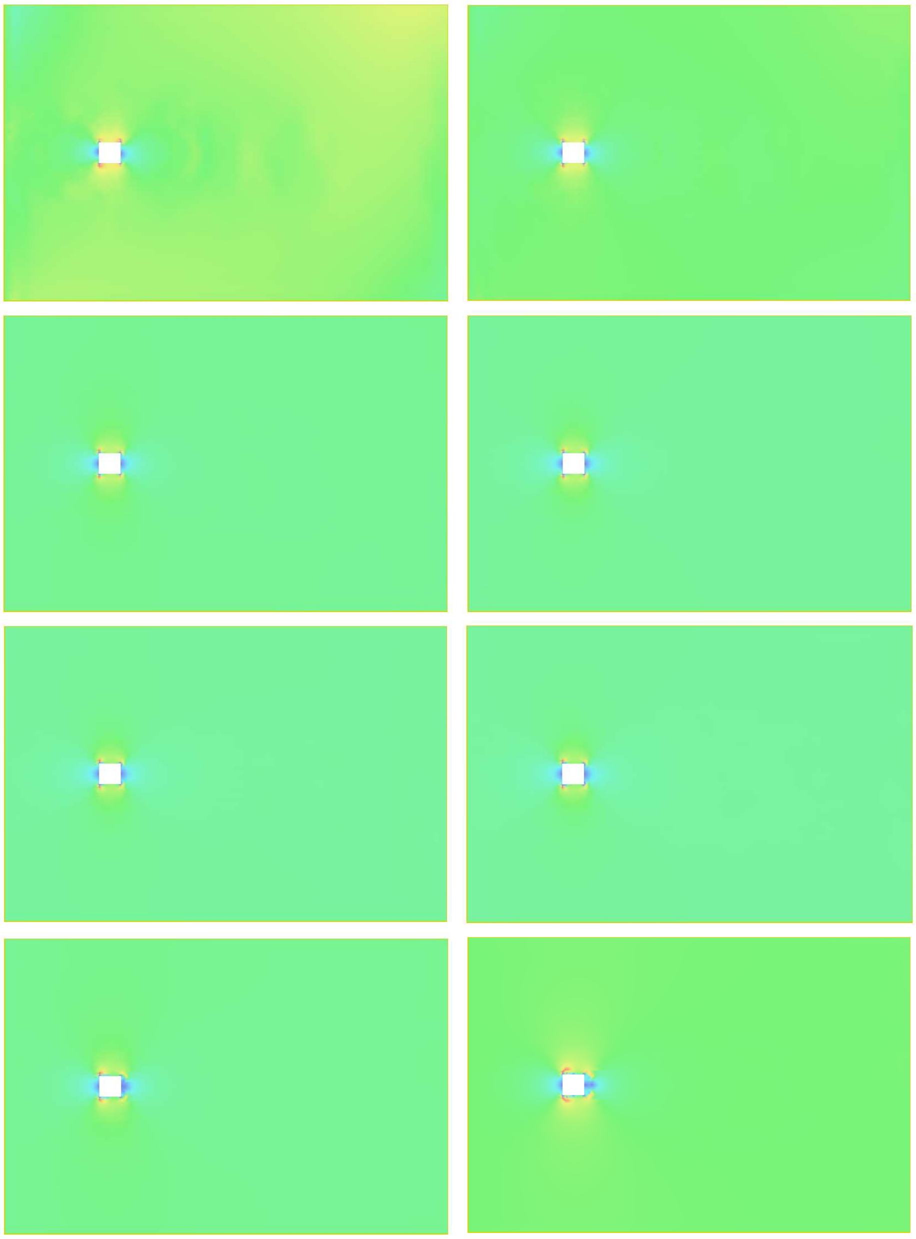

Figure 2. Magnitude of the computed velocity from initial data $U(0) = \nabla\varphi$, for time steps no 1, 2, 4, 5, 6, 7, 20, 37.

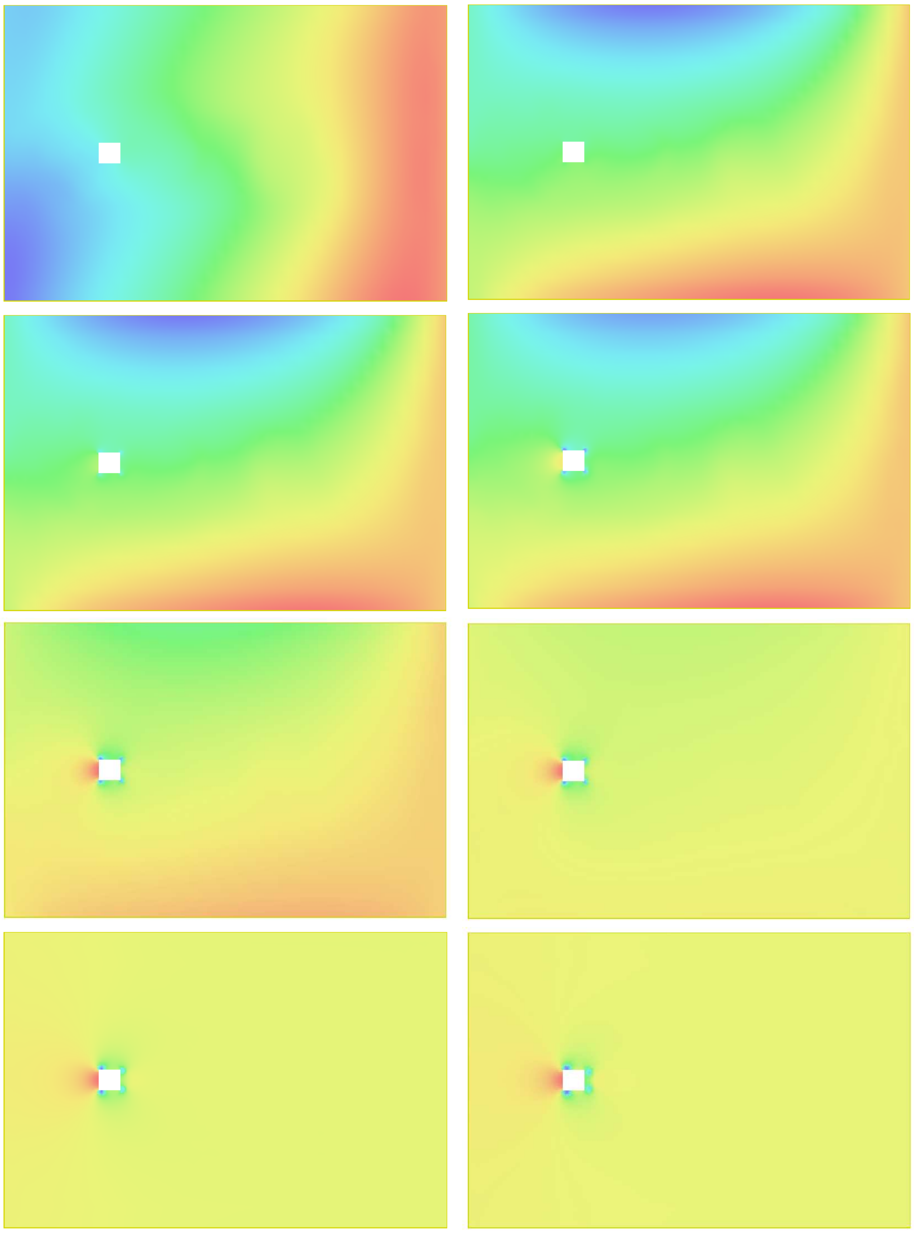

FIGURE 3. Computed pressure correspondig to the initial data $U(0) = \nabla\varphi$, for time steps no 1, 2, 4, 5, 6, 7, 20, 37.

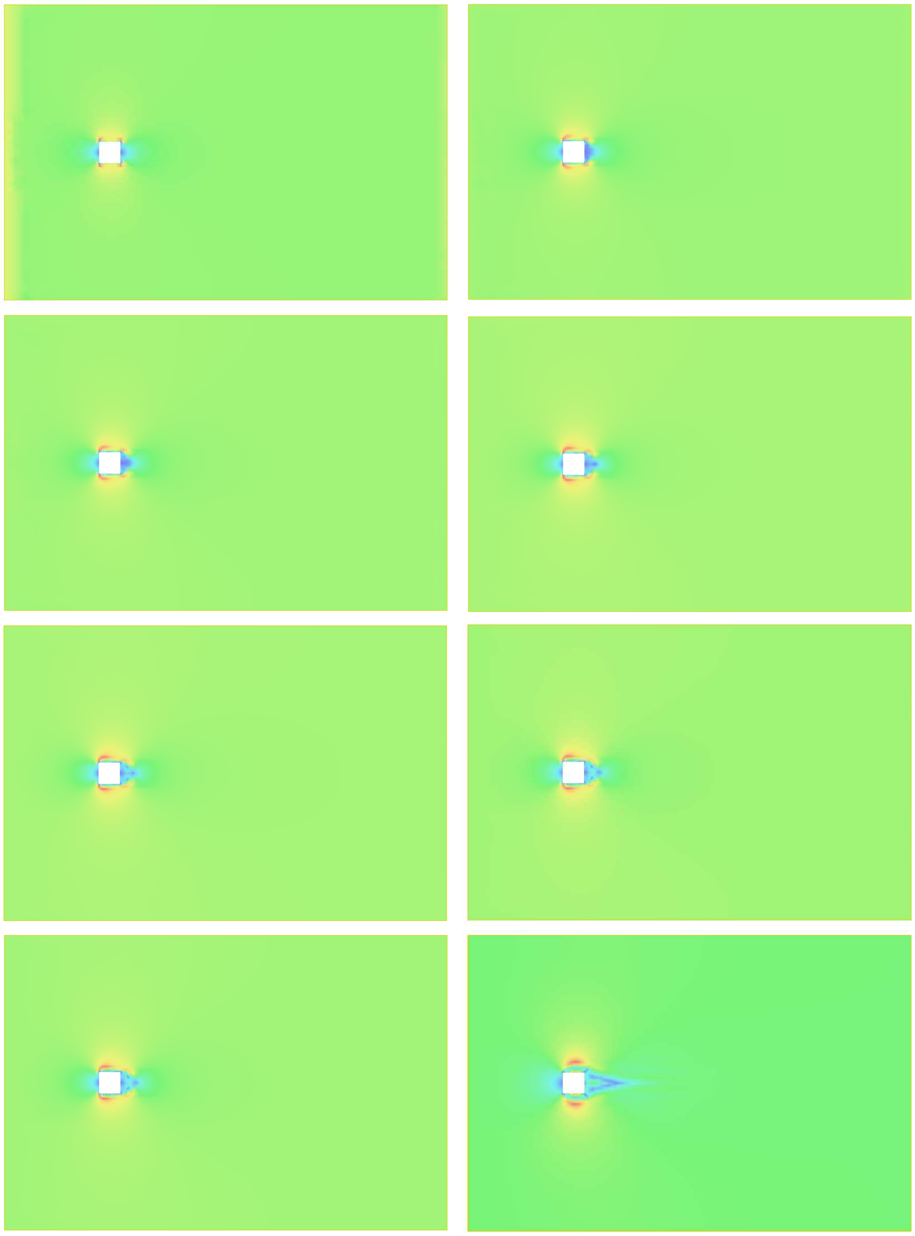

FIGURE 4. Magnitude of the computed velocity corresponding to zero initial data, for time steps 2, 4, 5, 6, 7, 8, 16, 32.

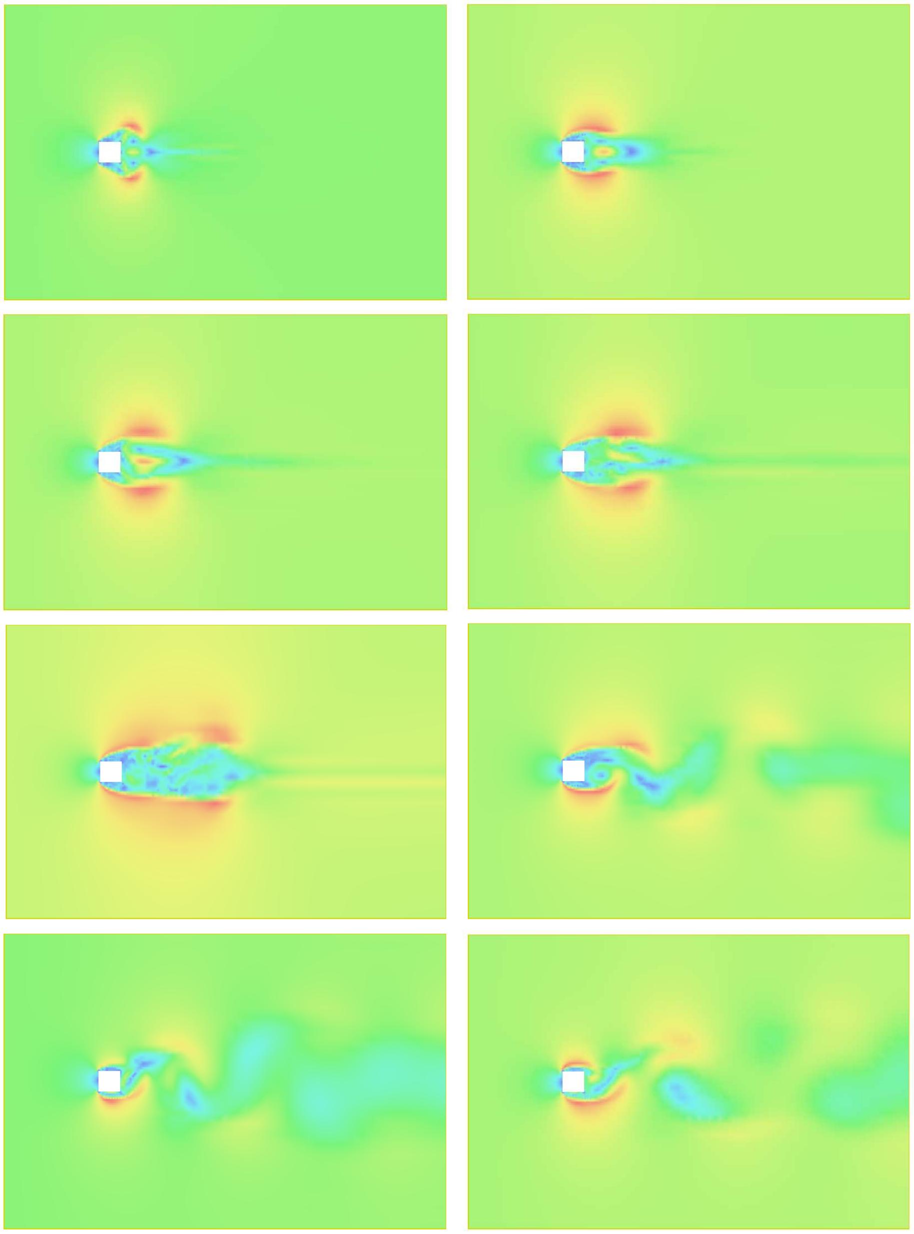

Figure 5. Magnitude of the computed velocity corresponding to zero initial data, for time $t = 0.75, 1, 1.5, 2, 2.5, 11, 15, 16$.

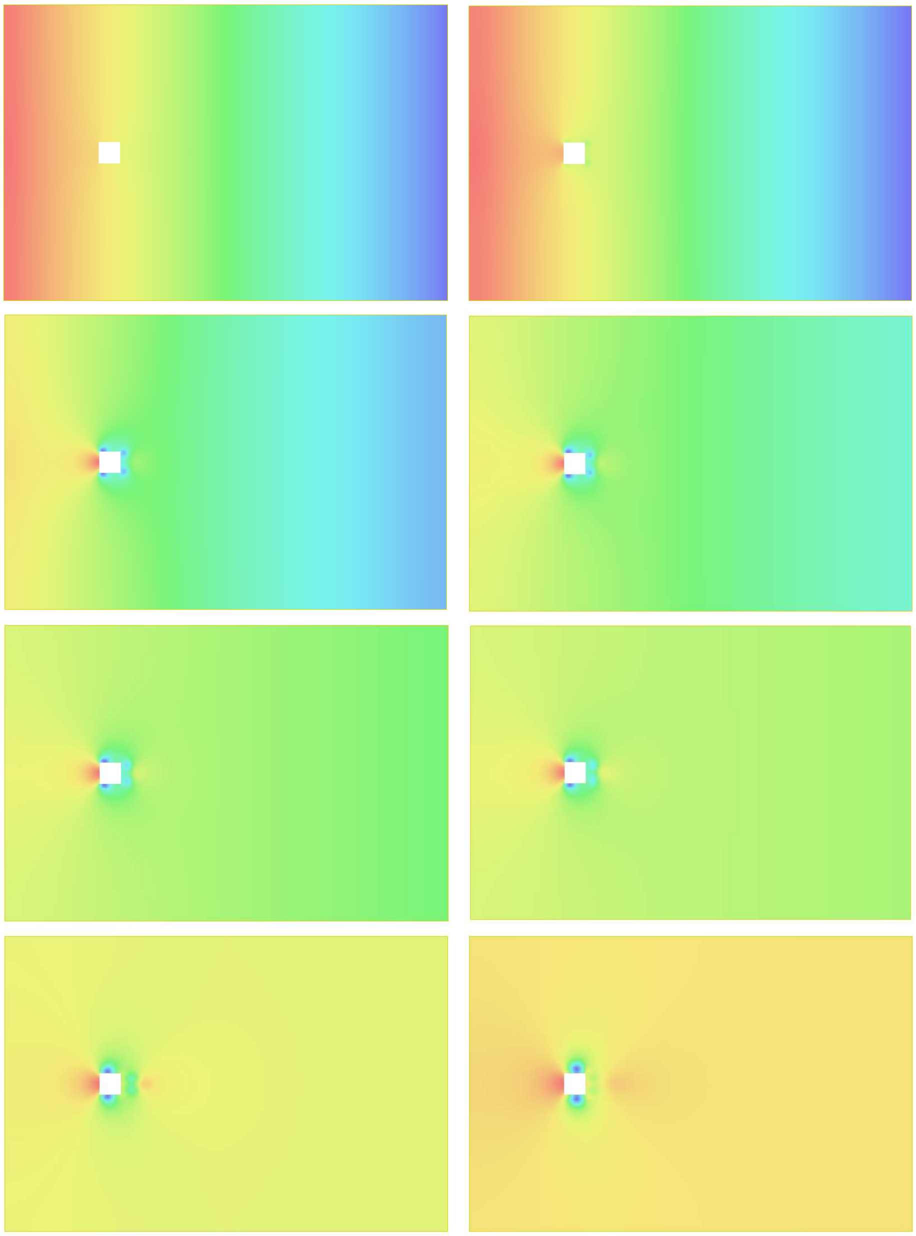

FIGURE 6. Pressure corresponding to zero initial data, for time steps 2, 4, 5, 6, 7, 8, 16, 32.

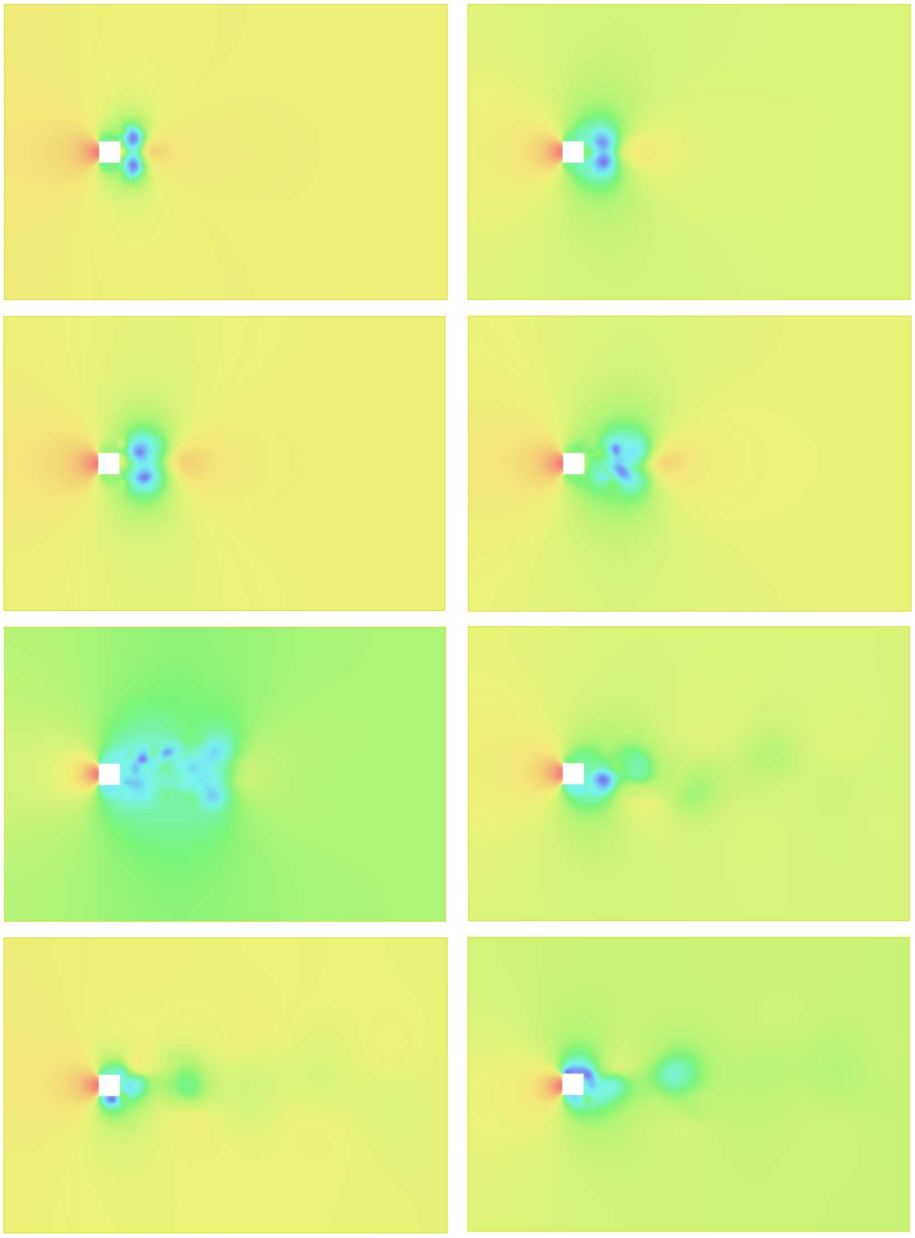

FIGURE 7. Pressure corresponding to zero initial data, for time $t = 0.75, 1, 1.5, 2, 2.5, 11, 15, 16$.

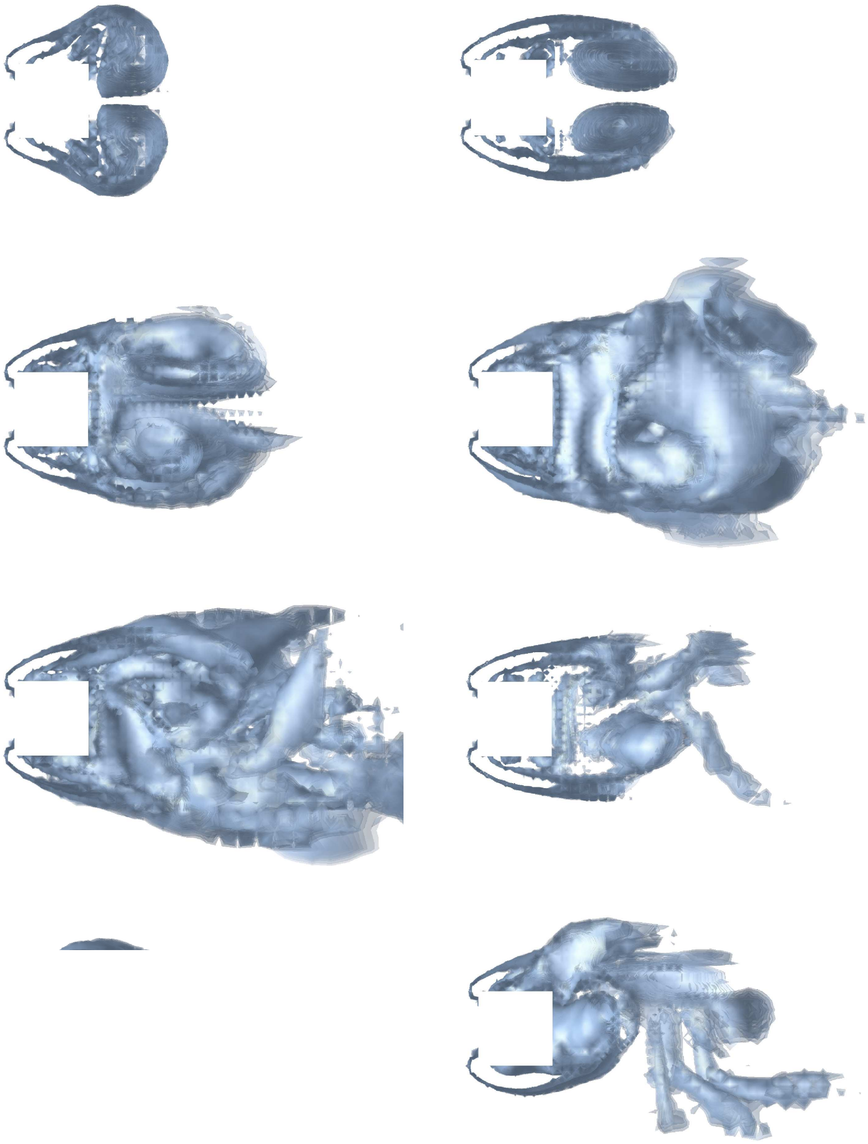

FIGURE 8. Isosurfaces for the magnitude of vorticity of the order 20-50, corresponding to zero initial data, for time $t = 0.75, 1, 1.5, 2, 2.5, 11, 15, 16$.

15. TEMPERATURE

> In a reasonable theory there are no dimensionless numbers whose values are only empirically determinable. (Einstein)

The total energy e is the sum of the kinetic energy and the internal energy i:

$$e = \frac{1}{2}|u|^2 + i, \tag{15.1}$$

where the internal energy $i = c_v\theta$ represents heat, which we assume to be proportional to the temperature θ with constant heat capacity c_v. Conservation of the total energy is expressed by the conservation law

$$\dot{e} + \nabla \cdot (eu + pu) = 0. \tag{15.2}$$

Having computed $\hat{u} = (u, p)$ from the incompressible Euler equations, we can solve for the total energy e in the linear equation (15.2) with u and p given, to obtain the internal energy/temperature from (15.1). In Fig. 9-10 we show the computed temperature starting from zero temperature at initial time and letting the inflow temperature be equal to zero. We notice that the temperature is elevated in the turbulent wake, with the heat being generated by the turbulent dissipation (represented by the weighted least squares term in G2). We notice that the generated heat is transported by the turbulent velocity u in a process of turbulent diffusion of heat, which most likely will dominate any molecular diffusion of heat (which we effectively set to zero in the computation). We are thus able to compute a temperature distribution in a turbulent flow with the only information that the coefficients of viscosity and molecular heat diffusion are very small. This is very good news since precise quantitative determination of very small viscosities or heat conductivitities is very difficult both theoretically and experimentally. From the computations we get the message that the precise values of these (small) quantities are irrelevant, if the quantities of interest are certain mean values.

We have been led to the conclusion that, up to the scaling of the temperature, the Euler equations may be used as a working model of thermodynamics, where no physical constants appear. This corresponds to a complete mathematization of a branch of physics, which thus does not require any input from experiments. Such a World is equal to the Euler equations solved by G2. The only parameter in this World is the mesh size h in G2, and the observable World seems to be almost independent of h if only h is small [4].

16. CONCLUSION

We have presented an example of a resolution of the classical paradox of irreversibility in formally reversible models, in the form of the Euler equations for incompressible inviscid flow. We used a computational approach based on a Generalized Galerkin G2 method computing a solution with residual being small in weak sense and with a weighted least squares control of the residual. We observed that the irreversibility arises because the Euler equations lack stable pointwise solutions and because the turbulent solutions which emerge in G2 computations necessarily and irreversibly loose energy in the dissipative least squares stabilization. The irreversibility in G2 is thus a necessary consequence of the the non-existence of stable pointwise solutions to the Euler equations. We conjecture that Nature has to handle the dilemma of non-existence in some similar form of analog computation. We continue our study in [6, 7].

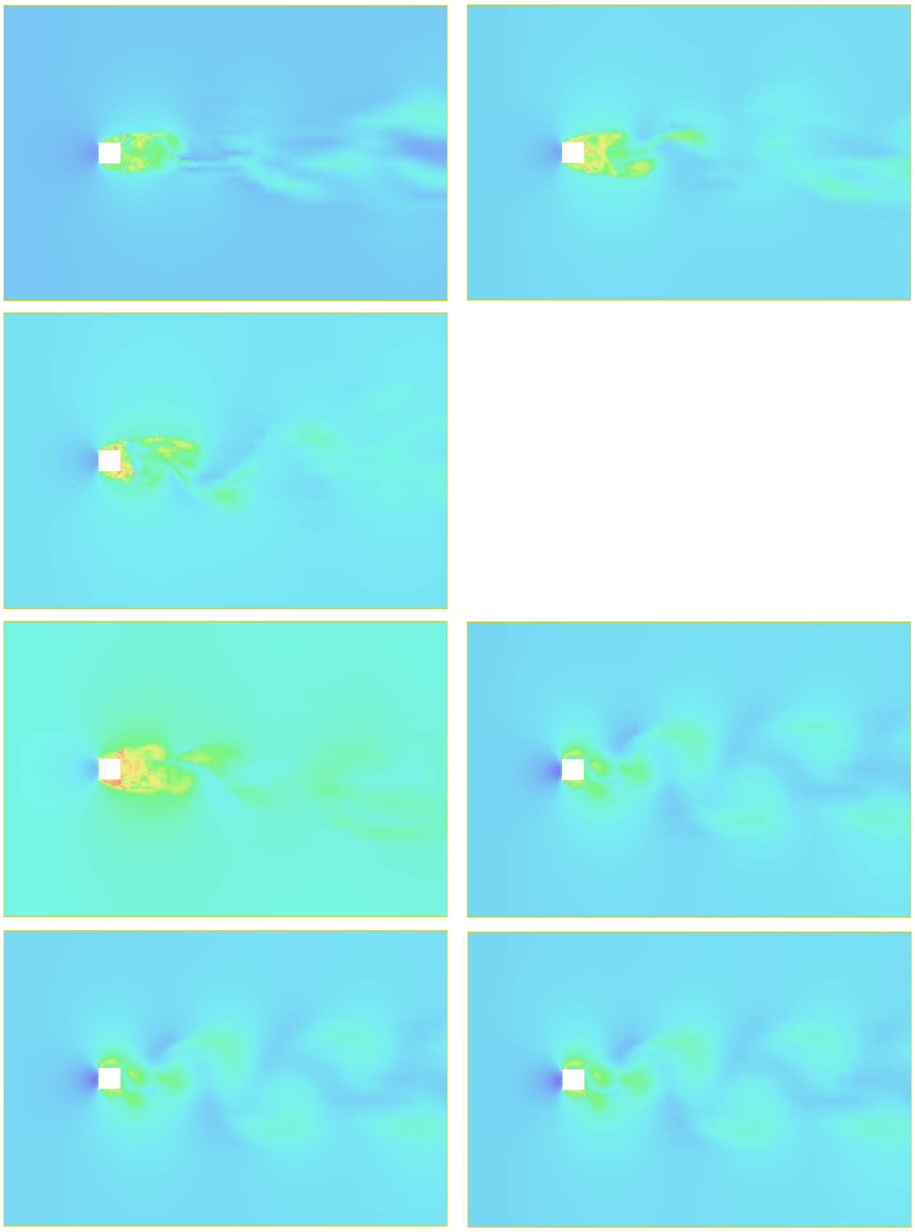

FIGURE 9. Total energy e, for time $t = 4, 4.5, 5, 5.5, 6, 11, 15, 16$.

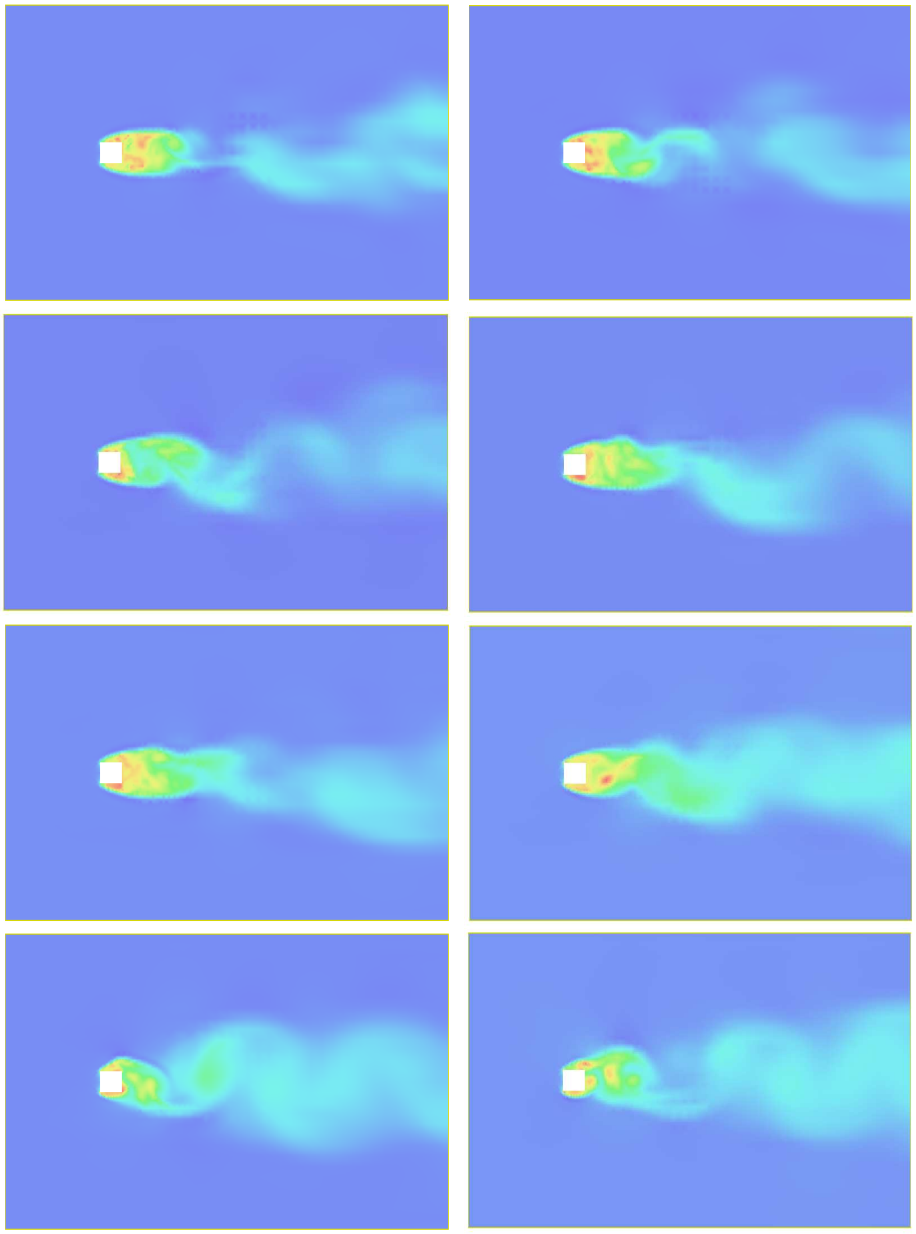

FIGURE 10. Internal energy $i = c_v\theta$, for time $t = 4, 4.5, 5, 5.5, 6, 11, 15, 16$.

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