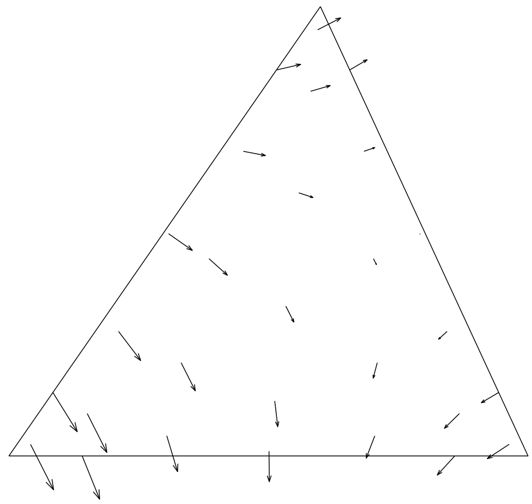
CHALMERS

FINITE ELEMENT CENTER



PREPRINT 2005–10

Piecewise divergence free discontinuous Galerkin methods

Peter Hansbo and Mats G. Larson

Chalmers Finite Element Center
CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg Sweden 2006

CHALMERS FINITE ELEMENT CENTER

Preprint 2005–10

Piecewise divergence free discontinuous Galerkin methods

Peter Hansbo and Mats G. Larson

CHALMERS

Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg Sweden
Göteborg, January 2006

Piecewise divergence free discontinuous Galerkin methods
Peter Hansbo and Mats G. Larson
NO 2005–10
ISSN 1404–4382

Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg
Sweden
Telephone: +46 (0)31 772 1000
Fax: +46 (0)31 772 3595
www.phi.chalmers.se

Printed in Sweden
Chalmers University of Technology
Göteborg, Sweden 2006

PIECEWISE DIVERGENCE FREE DISCONTINUOUS GALERKIN METHODS

PETER HANSBO AND MATS G. LARSON

Abstract. In this paper we consider different possibilities of using divergence free discontinuous Galerkin methods for Stokes in order to eliminate the pressure from the discrete problem. We focus on three different approaches: one based on a C^0 approximation of the the stream function in two dimensions (the vector potential in three dimensions), one based on the nonconforming Morley element (which corresponds to a divergence free nonconforming Crouzeix-Raviart approximation of the velocities), and one fully discontinuous Galerkin method with a stabilization of the pressure that allows the edgewise elimination of the pressure variable before solving the discrete system. We limit the analysis in the stream function case to two spatial dimensions, while the analysis of the fully discontinuous approach is valid also in three dimensions.

1. Introduction

In the finite element approximation of the Stokes model of incompressible flow, the incompressibility condition is usually imposed weakly by means of a Lagrange multiplier whose physical interpretation is that of the pressure in the fluid. Another option that has sometimes been proposed, e.g., by Thomasset [13] and by Hecht [9], is to directly construct divergence free bases, or, alternatively, indirectly by use a of stream function approach, which however leads to a higher order differential equation to be solved. A recent alternative to these approaches, using a hybridization technique, was given by Cockburn and Gopalakrishnan [4, 5].

If we directly apply the discontinuous Galerkin (DG) method to the Stokes system using an elementwise divergence free *ansatz*, we do not automatically get rid of the pressure since it is still needed to enforce normal continuity of the velocities. Our aim in this paper is to consider some different DG approaches that allow the complete elimination of the pressure from the system. We will consider three different ways of achieving this:

- To use a penalty method for controlling the jump in normal velocity. We then relate the strength of the penalty parameter so that the consistency error matches the discretization error.

Date: January 5, 2006.

Key words and phrases. solenoidal elements, Stokes problem, stream function, discontinuous Galerkin.

Peter Hansbo, Department of Applied Mechanics, Chalmers University of Technology, S–412 96 Göteborg, Sweden, *email*: hansbo@solid.chalmers.se

Mats G. Larson, Department of Mathematics and Mathematical Statistics, Umeå University, SE–901 87 Umeå, Sweden, *email*: mats.larson@math.umu.se.

- To use the curl of a conforming basis as a divergence free basis. This relates to the s tream function approach of Baker and Jureidini [1] but with the difference that we follow the approach of Johnson and Saranen [10] in formulating the problem in physical variables. As we shall see, the price for avoiding the direct use of the stream function is a consistency error on the boundary that we again control by penalty.
- To use the curl of the nonconforming Morley basis [11] as a divergence free basis (which will reside in the nonconforming Crouzeix-Raviart space [6]). This is related to the approach of [13], but without the explicit construction of the basis. The consistency error on the boundary is again handled by penalty.

In Section 2 we formulate the Stokes problem, define the finite element spaces, and formulate the Discontinuous Galerkin methods; in Section 3 we state and prove the basic analytical results including an error estimate in the energy norm; in Section 4 we show some numerical results for a smooth and a nonsmooth test problem.

2. Continuous and discrete problems

2.1. **The continuous problem.** We consider the Stokes problem: Find the velocity $\boldsymbol{u} = (u_i)_{i=1}^d$, with $d = 2$ or $d = 3$, and the pressure p such that

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{2.3}$$

where $\boldsymbol{f}$ is a given load, $\boldsymbol{g}$ is the given boundary velocity satisfying $(\boldsymbol{n}\cdot\boldsymbol{g}, 1)_{\partial\Omega} = 0$, and Ω is a simply connected domain in $\mathbf{R}^d$ with boundary $\partial\Omega$. When $\boldsymbol{f} \in H^{-1}(\Omega)$, $\boldsymbol{g} \in H^{1/2}(\Gamma)$ there exists a unique solution $(\boldsymbol{u}, p) \in H^1(\Omega) \times L^2_0(\Omega)$, see [7].

Next we define the curl operator $\nabla\times$ as follows

$$\nabla \times \varphi := \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right) \quad \text{for } d = 2, \tag{2.4}$$

$$\nabla \times \boldsymbol{\varphi} := \left(\frac{\partial \varphi_3}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_3}, \frac{\partial \varphi_1}{\partial x_3} - \frac{\partial \varphi_3}{\partial x_1}, \frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} \right) \quad \text{for } d = 3. \tag{2.5}$$

Since $\nabla \cdot \boldsymbol{u} = 0$ there is a unique function $\boldsymbol{\varphi} \in H(\nabla\times, \Omega) = \{\boldsymbol{v} \in L^2(\Omega) : \nabla \times \boldsymbol{v} \in L^2(\Omega)\}$ such that

$$\boldsymbol{u} = \nabla \times \boldsymbol{\varphi}, \tag{2.6}$$

where $\boldsymbol{\varphi}$ satisfies $\nabla \cdot \boldsymbol{\varphi} = 0$ and the following boundary conditions

$$\varphi = 0 \quad \text{for } d = 2, \tag{2.7}$$

$$\boldsymbol{n} \times \boldsymbol{\varphi} = 0 \quad \text{for } d = 3. \tag{2.8}$$

see [7] for further details.

2.2. **Finite element spaces.** Consider a subdivision of Ω into a geometrically conforming partitioning $\mathcal{T} = \{T\}$ of Ω into shape regular triangles in two dimensions and tetrahedra in three dimensions. On this subdivision we introduce the finite element space of discontinuous piecewise polynomials of degree $p+1$:

$$V^h = \{v : \, v|_T \in P^{p+1}(T), \quad \forall T \in \mathcal{T}\},$$

then we let the stream function space S^h satisfy

$$C(\Omega) \cap V^h \subset S^h \subset V^h, \quad \text{for } d = 2, \tag{2.9}$$

$$[C(\Omega) \cap V^h]^3 \subset \boldsymbol{S}^h \subset [V^h]^3, \quad \text{for } d = 3 \ , \tag{2.10}$$

and, finally, we define the velocity space by taking the elementwise curl of functions in the stream function space

$$\boldsymbol{W}^h = \{\boldsymbol{v} : \, \boldsymbol{v}|_T = \nabla \times \varphi, \ \varphi \in S^h\}, \quad \text{for } d = 2, \tag{2.11}$$

$$\boldsymbol{W}^h = \{\boldsymbol{v} : \, \boldsymbol{v}|_T = \nabla \times \boldsymbol{\varphi}, \ \boldsymbol{\varphi} \in \boldsymbol{S}^h\}, \quad \text{for } d = 3 \ . \tag{2.12}$$

We note that the velocity space consists of piecewise divergence free polynomials. When S^h is small we also have normal continuity of the velocity field, which is useful when extending the method from Stokes to Navier–Stokes equations, cf. [10].

In three dimensions, the use of a stream function is less attractive, due to it being defined only up to gradients of scalar fields (harmonic fields if an Euclidean gauge is applied). Thus we limit the analysis to the two-dimensional case for the stream function approach, while the analysis for the fully discontinuous *ansatz* holds without modifications also in three dimensions.

2.3. **The discrete problem.** To define the finite element method, let ∂T_I denote the faces of the element T neighboring to other elements, ∂T_D the faces on $\partial\Omega$, and let h denote the diameter of T. Further, for $\boldsymbol{x} \in \partial T_I$, let $[\boldsymbol{U}] = \boldsymbol{U}^+ - \boldsymbol{U}^-$ and $\langle \boldsymbol{U} \rangle = (\boldsymbol{U}^+ + \boldsymbol{U}^-)/2$, where

$$\boldsymbol{U}^\pm = \lim_{\epsilon \downarrow 0} \boldsymbol{U}(\boldsymbol{x} \mp \epsilon\, \boldsymbol{n}),$$

i.e., $\boldsymbol{U}^+$ belongs to T and $\boldsymbol{U}^-$ to its neighbor. For $\boldsymbol{x} \in \partial T_D$, we let $[\boldsymbol{U}] = \langle \boldsymbol{U} \rangle = \boldsymbol{U}^+$. On each face $E = \partial T^+ \cap \partial T^-$, the mesh parameter h is defined by

$$h := \frac{m(T^+) + m(T^-)}{2m(E)}, \tag{2.13}$$

where $m(\cdot)$ denotes the appropriate Lebesgue measure. To each edge E, we associate a fixed normal vector $\boldsymbol{n} := \boldsymbol{n}_{T^+}$.

The discontinuous Galerkin method then reads: find $\boldsymbol{U} \in \boldsymbol{W}^h$ such that

$$a_h(\boldsymbol{U}, \boldsymbol{v}) = l_h(\boldsymbol{v}), \tag{2.14}$$

for all $\boldsymbol{v} \in \boldsymbol{W}^h$, where

$$a_h(\boldsymbol{U},\boldsymbol{v}) = \sum_{T\in\mathcal{T}}(\nabla\boldsymbol{U},\nabla\boldsymbol{v})_T - (\langle\boldsymbol{n}\cdot\nabla\boldsymbol{U}\rangle,[\boldsymbol{v}])_{\partial T_I} - (\boldsymbol{n}\cdot\nabla\boldsymbol{U},\boldsymbol{v})_{\partial T_D} - ([\boldsymbol{U}],\langle\boldsymbol{n}\cdot\nabla\boldsymbol{v}\rangle)_{\partial T_I} - (\boldsymbol{U},\boldsymbol{n}\cdot\nabla\boldsymbol{v})_{\partial T_D} + \frac{1}{2}(\frac{\beta}{h}\left[\boldsymbol{U}\right],[\boldsymbol{v}])_{\partial T_I} + (\frac{\beta}{h}\boldsymbol{U},\boldsymbol{v})_{\partial T_D} + \frac{1}{2}(\frac{\gamma}{h^3}\left[\boldsymbol{n}\cdot\boldsymbol{U}\right],[\boldsymbol{n}\cdot\boldsymbol{v}])_{\partial T_I} + (\frac{\gamma}{h^3}\boldsymbol{n}\cdot\boldsymbol{U},\boldsymbol{n}\cdot\boldsymbol{v})_{\partial T_D}, \tag{2.15}$$

$$l_h(\boldsymbol{v}) = \sum_{T\in\mathcal{T}}(\boldsymbol{f},\boldsymbol{v})_T - (\boldsymbol{g},\boldsymbol{n}\cdot\nabla\boldsymbol{v})_{\partial T_D} + (\frac{\beta}{h}\boldsymbol{g},\boldsymbol{v})_{\partial T_D} + (\frac{\gamma}{h^3}\boldsymbol{n}\cdot\boldsymbol{g},\boldsymbol{n}\cdot\boldsymbol{v})_{\partial T_D}, \tag{2.16}$$

with $(v,w)=\int_\omega vw$, $(\boldsymbol{v},\boldsymbol{w})_\omega=\sum_{i=1}^d(v_i,w_i)$, and $(\nabla\boldsymbol{U},\nabla\boldsymbol{v})_T=\sum_{i=1}^d(\nabla U_i,\nabla v_i)_T$.

Remark 1. *The penalty term on the normal jumps can be motivated as follows. Let us define the following nonstandard discrete space for the pressures:*

$$Q^h = \{q\in L_2(\mathcal{E}):\ q|_E\in P^p(E)\},$$

the bilinear form

$$b_h(p,\boldsymbol{v}) := \sum_T\left(\frac{1}{2}(p,[\boldsymbol{n}\cdot\boldsymbol{v}])_{\partial T_I} + (p,\boldsymbol{n}\cdot\boldsymbol{v})_{\partial T\cap\partial\Omega}\right)$$

and consider the following consistent *finite element method: find $(\boldsymbol{U},P)\in\boldsymbol{W}^h\times Q^h$ such that*

$$a_h(\boldsymbol{U},\boldsymbol{v}) - b_h(P,\boldsymbol{v}) + b_h(q,\boldsymbol{U}) = L_h(\boldsymbol{v}),\quad \forall(\boldsymbol{v},q)\in\boldsymbol{W}^h\times Q^h. \tag{2.17}$$

The stability of this method is an issue in itself, but we now consider modifying the discrete problem by adding an inconsistent stabilizing term to obtain the problem of finding $(\boldsymbol{U},P)\in\boldsymbol{W}^h\times Q^h$ such that

$$a_h(\boldsymbol{U},\boldsymbol{v}) - b_h(P,\boldsymbol{v}) + b_h(q,\boldsymbol{U}) - \sum_{E\in\mathcal{E}}h^3(P,q)_E = L_h(\boldsymbol{v}),\quad \forall(\boldsymbol{v},q)\in\boldsymbol{W}^h\times Q^h. \tag{2.18}$$

This allows for the direct elimination of the pressure; since $[\boldsymbol{n}\cdot\boldsymbol{U}]|_E\in P^p(E)$ we immediately obtain

$$P|_E := h^{-3}[\boldsymbol{n}\cdot\boldsymbol{U}],$$

and we regain the method (2.14).

3. Error estimate

The method (2.14) is not consistent in general. Instead we may use Green's formula to derive the following modified orthogonality relation.

Proposition 1. *Let $\boldsymbol{u}, p$ be the solution to (2.1) and $\boldsymbol{U}$ the solution to (2.14). Then the following identity holds*

$$a_h(\boldsymbol{u}-\boldsymbol{U},\boldsymbol{v}) = \sum_{T\in\mathcal{T}} \Big(\frac{1}{2}(p,[\boldsymbol{n}\cdot\boldsymbol{v}])_{\partial T_I} + (p,\boldsymbol{n}\cdot\boldsymbol{v})_{\partial T_D}\Big),$$

for all $\boldsymbol{v}\in\boldsymbol{W}^h$ and for $\boldsymbol{u}$ and p sufficiently regular.

We thus note that the method is consistent when the functions in $\boldsymbol{W}^h$ have continuous normal components. Consequently, the stream function approach with a C^0 approximation is consistent if we apply the boundary conditions strongly on the stream function itself. This is indeed possible, and not too technically difficult in two dimensions even for multi-connected domains, cf. the discussion in [12]. Here we take the alternative approach of imposing prescribed velocities weakly on the boundary, which is technically easier to implement. The price to pay is that we must then retain the $O(h^{-3})$ penalty term on the boundary. We first consider the cases of the full DG and of the C^0 stream function approximation. The nonconforming stream function approach is discussed in Remark 2.

We shall measure the error in the following mesh dependent norm

$$\begin{aligned}|||\boldsymbol{v}|||^2 = \ &\sum_{T\in\mathcal{T}} \|\nabla\boldsymbol{v}\|^2_T + \|h^{1/2}\langle\boldsymbol{n}\cdot\nabla\boldsymbol{v}\rangle\|^2_{\partial T_I} + \|h^{1/2}\boldsymbol{n}\cdot\nabla\boldsymbol{v}\|^2_{\partial T_{\mathrm{D}}} \\ &+\|h^{-1/2}[\boldsymbol{v}]\|^2_{\partial T_I} + \|h^{-1/2}[\boldsymbol{v}]\|^2_{T_{\partial \mathrm{D}}} \\ &+\|h^{-3/2}[\boldsymbol{n}\cdot\boldsymbol{v}]\|^2_{\partial T_I} + \|h^{-3/2}\boldsymbol{n}\cdot\boldsymbol{v}\|^2_{\partial T_D},\end{aligned} \tag{3.1}$$

where $\|w\|^2_\omega = \int_\omega w^2$ is the $L^2(\omega)$-norm. With respect to the norm (3.1) we have the following coercivity and continuity result which can be proved using standard arguments, see [8] for details.

Proposition 2. *If β is large enough and $\gamma\geq 0$ then the bilinear form is coercive, i.e.,*

$$m|||\boldsymbol{v}|||^2 \leq a_h(\boldsymbol{v},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v}\in\boldsymbol{W}^h, \tag{3.2}$$

for some constant $m>0$ independent of the meshsize h. Furthermore, we the bilinear form is continuous

$$a_h(\boldsymbol{v},\boldsymbol{w}) \leq |||\boldsymbol{v}|||\,|||\boldsymbol{w}||| \quad \text{for all } \boldsymbol{v},\boldsymbol{w}\in\left[H^1(\Omega)\right]^d + \boldsymbol{W}^h. \tag{3.3}$$

The exact convergence properties will depend on whether or not the approximation contains the Brezzi-Douglas-Marini space BDM_0 of full polynomial approximations with normal continuity and zero elementwise divergence (cf. [2] for the definition of BDM).

Proposition 3. *The following a priori error estimates hold:*

- *If* $\varphi \in H^{2+\alpha}$ *then*

$$|||\boldsymbol{u}-\boldsymbol{U}||| \leq C\left(h^{\alpha}|\varphi|_{2+\alpha}+h\|p\|_{1}\right), \tag{3.4}$$

with $0 \leq \alpha \leq p-1$, *if* $\varphi \in H^{2+\alpha}$.

- *If* $\mathrm{BDM}_0 \subseteq \boldsymbol{W}^h$ *and* $\boldsymbol{u} \in H^{1+\alpha}$ *then*

$$|||\boldsymbol{u}-\boldsymbol{U}||| \leq C\left(h^{\alpha}|\boldsymbol{u}|_{1+\alpha}+h\|p\|_{1}\right), \tag{3.5}$$

with $0 \leq \alpha \leq p$.

Proof. We first split the error $\boldsymbol{u}-\boldsymbol{U}=(\boldsymbol{u}-\boldsymbol{V})+(\boldsymbol{V}-\boldsymbol{U})$, where $\boldsymbol{V}$ an arbitrary function in $\boldsymbol{W}^h$. Using the triangle inequality we have

$$|||\boldsymbol{u}-\boldsymbol{U}||| = |||\boldsymbol{u}-\boldsymbol{V}||| + |||\boldsymbol{V}-\boldsymbol{U}|||. \tag{3.6}$$

Here the first term is an interpolation error term. To estimate the second term we employ the coercivity estimate (3.2) in Proposition 2, followed by the Galerkin orthogonality in Proposition 1, to get

$$\begin{aligned}
m|||\boldsymbol{V}-\boldsymbol{U}|||^2 &\leq a_h(\boldsymbol{V}-\boldsymbol{U},\boldsymbol{V}-\boldsymbol{U}) && (3.7)\\
&\leq a_h(\boldsymbol{V}-\boldsymbol{u},\boldsymbol{V}-\boldsymbol{U})+a_h(\boldsymbol{u}-\boldsymbol{U},\boldsymbol{V}-\boldsymbol{U}) && (3.8)\\
&\leq a_h(\boldsymbol{V}-\boldsymbol{u},\boldsymbol{V}-\boldsymbol{U}) && (3.9)\\
&\quad+\sum_{T\in\mathcal{T}}\frac{1}{2}(p,[\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U})])_{\partial T_I}+(p,\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U}))_{\partial T_D}.
\end{aligned}$$

We proceed with estimates of the two terms on the right hand side beginning with the second term. Using the Cauchy Schwartz inequality followed by a trace inequality we obtain

$$\begin{aligned}
(p,[\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U})])_{\partial T_I} &\leq \|p\|_{\partial T_I}\|[\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U})]\|_{\partial T_I}\\
&\leq C\Big(h^{-1}\|p\|_T^2+h\|\nabla p\|_T^2\Big)^{1/2}\|[\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U})]\|_{\partial T_I} && (3.10)\\
&\leq \frac{C^2h^3}{\epsilon}\Big(h^{-1}\|p\|_T^2+h\|\nabla p\|_T^2\Big)+\frac{\epsilon}{4h^3}\|[\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U})]\|_{\partial T_I}^2,
\end{aligned}$$

where ϵ is at our disposal. Now we can use kickback on the last term and estimate the first term as follows:

$$\sum_{T\in\mathcal{T}} h^2(\|p\|_T^2+h^2\|\nabla p\|_T^2) \leq h^2\|p\|_1^2. \tag{3.11}$$

The boundary term is handled in the same way. Next, we turn to the first term on the right hand side of (3.9). Using standard arguments we have

$$a_h(\boldsymbol{V}-\boldsymbol{u},\boldsymbol{V}-\boldsymbol{U}) \leq \frac{C}{\epsilon}|||\boldsymbol{V}-\boldsymbol{u}|||^2+\frac{\epsilon}{4}|||\boldsymbol{V}-\boldsymbol{U}|||^2, \tag{3.12}$$

where we can use kickback on the last term. Collecting the estimates (3.6), (3.9), (3.10), and (3.12) we arrive at

$$|||\boldsymbol{u}-\boldsymbol{U}|||^2 \leq C(|||\boldsymbol{u}-\boldsymbol{V}|||^2+h^2\|p\|_1), \tag{3.13}$$

and it thus remains to estimate $|||\boldsymbol{u}-\boldsymbol{V}|||$. Using the trace inequality $\|w\|_{\partial T} \leq C\|w\|_{T,1/2}$, we obtain the following estimate

$$\begin{aligned}|||\boldsymbol{u}-\boldsymbol{V}|||^2 \leq C \sum_{T\in\mathcal{T}} &\|\boldsymbol{u}-\boldsymbol{V}\|^2_{1,T} + h\|\boldsymbol{u}-\boldsymbol{V}\|^2_{T,3/2} + h^{-1}\|\boldsymbol{u}-\boldsymbol{V}\|^2_{T,1/2} \\ &+ h^{-3}\|[\boldsymbol{n}\cdot(\boldsymbol{u}-\boldsymbol{V})]\|^2_{\partial T_I,1/2} + h^{-3}\|\boldsymbol{n}\cdot(\boldsymbol{u}-\boldsymbol{V})\|^2_{\partial T_D,1/2}.\end{aligned} \tag{3.14}$$

To prove the first estimate (3.4) we choose $\boldsymbol{V} = \nabla\times\pi_{\mathrm{SZ}}\boldsymbol{\varphi}$, with π_{SZ} the standard Scott-Zhang interpolation operator (cf. [3]) modified so that the boundary conditions (2.7) or (2.8) hold depending on the dimension. From the fact that the interpolant satisfies the boundary conditions it follows that $\boldsymbol{n}\cdot\nabla_{SZ}\boldsymbol{\varphi} = 0$ as well and thus the contribution from the normal trace vanishes at the boundary. Furthermore, the continuity of $\pi_{SZ}\boldsymbol{\varphi}$ implies that $\nabla\times\pi_{SZ}\boldsymbol{\varphi}$ has a continuous normal component. Using the standard interpolation error estimates we then get

$$|||\nabla\times\boldsymbol{\varphi} - \nabla\times\pi_{\mathrm{SZ}}\boldsymbol{\varphi}||| \leq Ch^{\alpha}|\boldsymbol{\varphi}|_{2+\alpha}, \tag{3.15}$$

which together with (3.13) proves (3.4). When $\boldsymbol{W}^h$ is large enough, so that $\mathrm{BDM}_0 \subseteq \boldsymbol{W}^h$ we can instead chose $\boldsymbol{V} = \pi_{\mathrm{BDM}}\boldsymbol{u}$. Starting from (3.14) and using standard interpolation error estimates, see [2], together with the fact that the BDM interpolant also has a vanishing normal trace we obtain the estimate

$$|||\boldsymbol{u} - \pi_{\mathrm{BDM}}\boldsymbol{u}||| \leq Ch^{\alpha}|\boldsymbol{u}|_{1+\alpha}, \tag{3.16}$$

which proves (3.5). □

Remark 2. *For the nonconforming Morley approximation of the stream function, the norm in which the problem is analyzed must be modified, since it is not possible to obtain full normal continuity on the boundary, and we face the difficulty of the interpolation in the terms*

$$\sum_T \|h^{-3/2}[\boldsymbol{n}\cdot\boldsymbol{u}]\|^2_{\partial T_I} + \sum_T \|h^{-3/2}\boldsymbol{n}\cdot\boldsymbol{u}\|^2_{\partial T_D},$$

which cannot be done without destroying the error estimate. For the internal faces, the corresponding term can be dropped from the norm. Since the curl of the Morley approximation resides in the Crouzeix-Raviart space (cf. Brenner [3]*), the velocities will have zero mean jump on the edges of the elements, and we can modify (3.10) to*

$$\begin{aligned}(p,[\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U})])_{\partial T_I} &= (p-\pi_0 p,[\boldsymbol{n}\cdot(\boldsymbol{V}-\boldsymbol{U})])_{\partial T_I} \\ &\leq C\|h^{1/2}(p-\pi_0 p)\|_{\partial T_I}\|h^{-1/2}[\boldsymbol{V}-\boldsymbol{U}]\|_{\partial T_I} \\ &\leq Ch\|\nabla p\|_T\|h^{-1/2}[\boldsymbol{V}-\boldsymbol{U}]\|_{\partial T_I} \\ &\leq \frac{C^2 h}{2\epsilon}\|\nabla p\|^2_T + \frac{\epsilon}{2}\|h^{-1/2}[\boldsymbol{V}-\boldsymbol{U}]\|^2_{\partial T_I},\end{aligned} \tag{3.17}$$

where π_0 is the projection onto constants on ∂T and ϵ is at our disposal. We can then proceed as above without the stronger interior penalty term. On the boundary, we must *instead replace the strong penalty by a weakened version: we modify the penalty terms in (2.15) and (2.16) to*

$$(\frac{\gamma}{h^3}\pi_0\boldsymbol{n}\cdot\boldsymbol{U},\pi_0\boldsymbol{n}\cdot\boldsymbol{v})_{\partial T_D} \quad \textit{and} \quad (\frac{\gamma}{h^3}\pi_0\boldsymbol{n}\cdot\boldsymbol{g},\pi_0\boldsymbol{n}\cdot\boldsymbol{v})_{\partial T_D},$$

respectively, and define

$$\begin{aligned}|||\boldsymbol{U}|||_*^2 := \sum_{T\in\mathcal{T}} &\|\nabla\boldsymbol{U}\|_T^2 + \|h^{1/2}\langle\boldsymbol{n}\cdot\nabla\boldsymbol{U}\rangle\|_{\partial T_I}^2 + \|h^{1/2}\boldsymbol{n}\cdot\nabla\boldsymbol{U}\|_{\partial T_D}^2 \\ &+\|h^{-1/2}[\boldsymbol{U}]\|_{\partial T_I}^2 + \|h^{-1/2}[\boldsymbol{U}]\|_{T_{\partial D}}^2 + \|h^{-3/2}\pi_0\boldsymbol{n}\cdot\boldsymbol{U}\|_{\partial T_D}^2,\end{aligned} \tag{3.18}$$

for which Proposition 2 still holds. It is then straightforward to find a Morley interpolant such that the interpolation error in the last term vanishes and, using the interpolation estimates from [11]*, we obtain the following a priori estimate: If* $\varphi \in H^4(\Omega)$*, then*

$$|||\boldsymbol{u}-\boldsymbol{U}|||_* \leq Ch\left(\|\varphi\|_3 + h\|\varphi\|_4 + \|p\|_1\right). \tag{3.19}$$

4. Numerical examples

In this section we give some numerical examples in two dimensions. In all examples we used $\gamma = 10$. The stream function is determined up to a constant, which was set by enforcing zero mean value for the stream function.

To give an impression of the vector bases implied by the P^2 and the Morley approximations, we give, in Figs. 1, 2, the appearance of a corner (lower left corner) and edge (bottom edge) basis function. Note the pure shear and curl appearance of the vector basis.

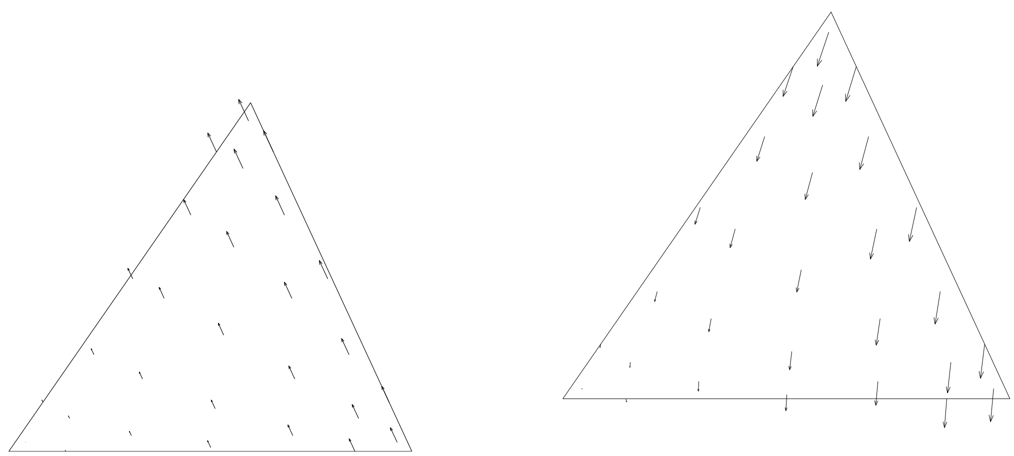


Figure 1. Implied basis vectors for a corner (left) and an edge (right) for the P^2–approximation.

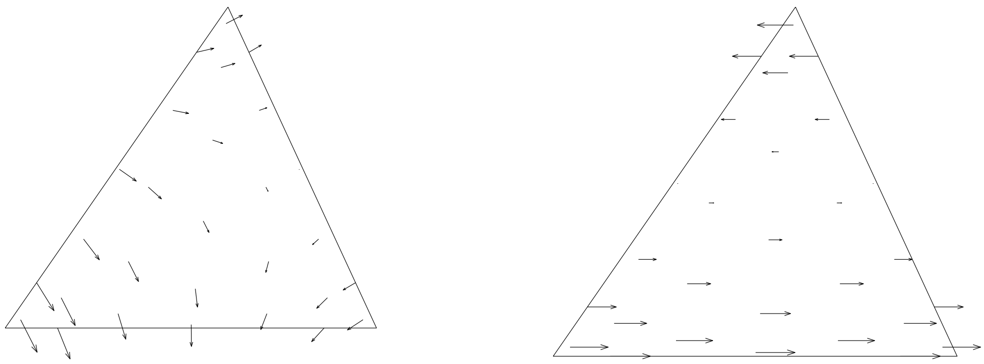


FIGURE 2. Implied basis vectors for a corner (left) and an edge (right) for the Morley approximation.

4.1. **A smooth problem.** We consider the unit square with exact flow solution given by $u = (20\,x\,y^3,\ 5\,x^4 - 5\,y^4)$. The problem is driven by boundary data, $\boldsymbol{f} = 0$. Isolines of the corresponding stream function are given in Fig. 3

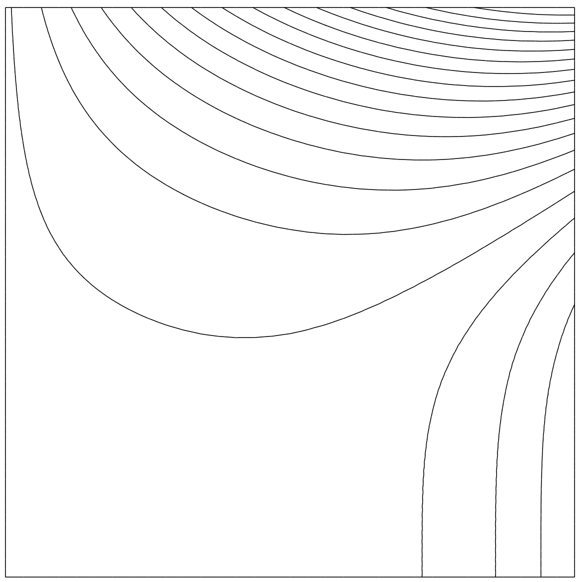


FIGURE 3. Smooth stream function.

In Figure 4 we compare the convergence obtained with the three methods in $L_2(\Omega)$ and in the broken H^1–norm. We note that the methods give the same optimal behaviour and are rather close in accuracy on a fixed mesh.

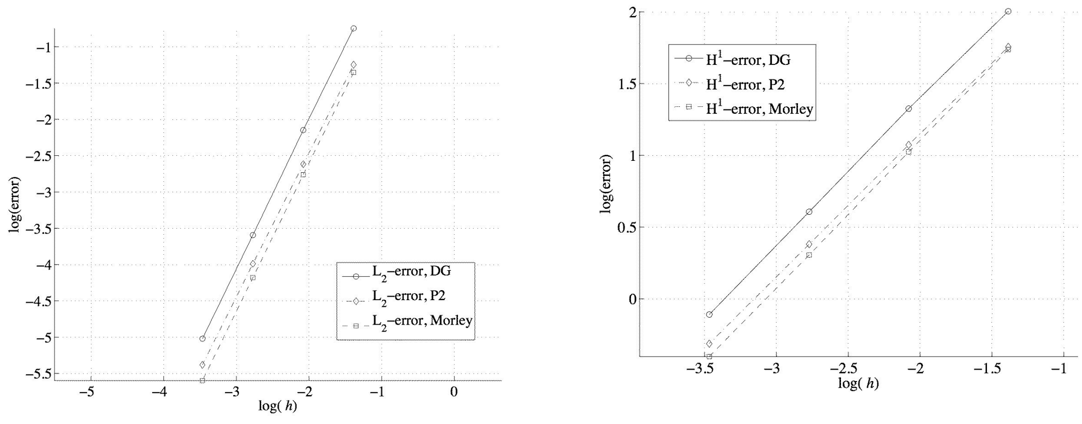


FIGURE 4. Convergence for the smooth problem.

4.2. **A non-smooth problem.** We consider a problem on an L-shaped domain (from Verfürth [14]) with exact solution given, in polar coordinates, by

$$\boldsymbol{u}(r,\phi) = r^{\lambda}\left[\begin{array}{c}(1+\lambda)\sin(\phi)\psi(\phi)+\cos(\phi)\psi'(\phi)\\ \sin(\phi)\psi'(\phi)-(1+\lambda)\cos(\phi)\psi(\phi)\end{array}\right],$$

where

$$\begin{aligned}\psi(\phi) \;&=\; \sin((1+\lambda)\phi)\cos(\lambda\omega)/(1+\lambda)-\cos((1+\lambda)\phi)\\ &\quad -\sin((1-\lambda)\phi)\cos(\lambda\omega)/(1-\lambda)+\cos((1-\lambda)\phi),\end{aligned}$$

$\omega = 3\pi/2$ and λ is the smallest positive root to

$$\sin(\lambda\omega)+\lambda\sin(\omega)=0,$$

yielding $\lambda \approx 0.544483736782 46$. Again, the problem is driven by boundary data, $\boldsymbol{f}=0$. For this problem $\boldsymbol{u}\notin H^2(\Omega)$, and we expect a corresponding decrease in convergence. In Fig. 5 we shw the isolines of the corresponding stream function, and in Fig. **??** we show the $L_2(\Omega)$–convergence of the different methods. Note that we now obtain only first order convergence due to the singularity at the reentrant corner.

5. CONCLUDING REMARKS

We have analyzed discontinuous Galerkin methods for the incompressible Stokes system using stream functions, where normal continuity of the *ansatz* is guaranteed, and a fully discontinuous approach, where normal continuity must be enforced using a stronger penalty in order to control the consistency error. Optimal convergence is observed for all the proposed methods.

In the case of a stream function approach, it is well known that the condition number is increased to $O(h^{-4})$ as compared to $O(h^{-2})$ for the standard FEM applied to second order partial differential equations. Interestingly, this effect cannot be avoided in the DG

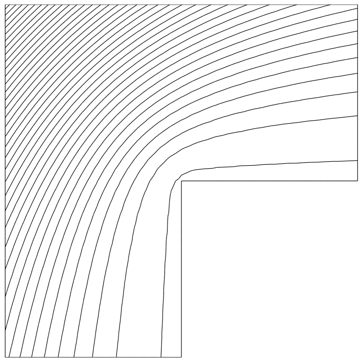


Figure 5. Stream function in the non-smooth case.

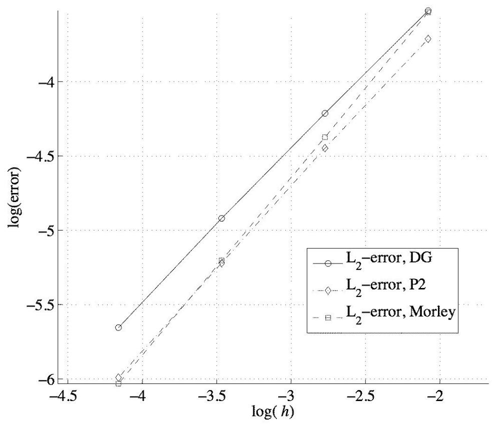


Figure 6. Convergence in a non-smooth case.

setting: the penalty necessary to control the consistency error for a fully discontinuous piecewise linear *ansatz* in order to retain the convergence properties of linear elements is of $O(h^{-3})$ which corresponds to a condition number of $O(h^{-4})$ for the system matrix. Thus, nothing can be gained with respect to conditioning. On the other hand, the fully discontinuous approach is much easier to implement than the stream function approach in three dimensions.

References

[1] Baker GA, Jureidini WN. A nonstandard finite element method for the stationary two-dimensional Navier–Stokes equations. *Computers & Mathematics with Applications* 1987; **13**(4):387–400.

[2] Brezzi F, Fortin M. *Mixed and Hybrid Finite Element Methods*, Springer–Verlag, 1991.

[3] Brenner S, Scott RL. *The mathematical theory of finite element methods. Second edition.* Springer–Verlag, 2002.

[4] Cockburn B, Gopalakrishnan J. Incompressible finite elements via hybridization. Part I: The Stokes system in two space dimensions. *SIAM Journal on Numerical Analysis* 2005; **43**(4):1627–1650.

[5] Cockburn B, Gopalakrishnan J. Incompressible finite elements via hybridization. Part II: The Stokes system in three space dimensions. *SIAM Journal on Numerical Analysis* 2005; **43**(4):1651–1672.

[6] Crouzeix M, Raviart PA. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. *RAIRO Série Rouge* 1973; **7**(R-3):33–75.

[7] Girault V, Raviart PA. *Finite Element Approximation of the Navier-Stokes Equations.* Springer: Berlin, 1986.

[8] Hansbo P, Larson MG. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Computer Methods in Applied Mechanics and Engineering* 2002; **191**(17-18):1895–1908.

[9] Hecht F. Construction d'une base de fonctions P_1 non conforme à divergence nulle dans R^3. *RAIRO Analyse Numérique* 1981; **15**(2):119–150.

[10] Johnson C, Saranen J. Streamline diffusion methods for the incompressible Euler and Navier–Stokes equations. *Mathematics of Computation* 1986; **47**(175):1–18.

[11] Lascaux P, Lesaint P. Some nonconforming finite elements for the plate bending problem. *RAIRO Analyse Numérique* 1975; **9**(R-1):9–53.

[12] Mizukami A. A stream function–vorticity finite element formulation for Navier-Stokes equations in multi-connected domain. *International Journal for Numerical Methods in Engineering* 1983; **19**(9):1403–1420.

[13] Thomasset F. *Implementation of Finite Element Methods for Navier-Stokes Equations,* Springer-Verlag, New York, 1981.

[14] Verfürth R. *A Review of* A Posteriori *Error Estimation and Adaptive Mesh–Refinement Techniques,* Wiley, 1996.

Chalmers Finite Element Center Preprints

2003–01 *A hybrid method for elastic waves*
Larisa Beilina

2003–02 *Application of the local nonobtuse tetrahedral refinement techniques near Fichera-like corners*
L. Beilina, S. Korotov and M. Křížek

2003–03 *Nitsche's method for coupling non-matching meshes in fluid-structure vibration problems*
Peter Hansbo and Joakim Hermansson

2003–04 *Crouzeix–Raviart and Raviart–Thomas elements for acoustic fluid–structure interaction*
Joakim Hermansson

2003–05 *Smoothing properties and approximation of time derivatives in multistep backward difference methods for linear parabolic equations*
Yubin Yan

2003–06 *Postprocessing the finite element method for semilinear parabolic problems*
Yubin Yan

2003–07 *The finite element method for a linear stochastic parabolic partial differential equation driven by additive noise*
Yubin Yan

2003–08 *A finite element method for a nonlinear stochastic parabolic equation*
Yubin Yan

2003–09 *A finite element method for the simulation of strong and weak discontinuities in elasticity*
Anita Hansbo and Peter Hansbo

2003–10 *Generalized Green's functions and the effective domain of influence*
Donald Estep, Michael Holst, and Mats G. Larson

2003–11 *Adaptive finite element/difference method for inverse elastic scattering waves*
Larisa Beilina

2003–12 *A Lagrange multiplier method for the finite element solution of elliptic domain decomposition problems using non-matching meshes*
Peter Hansbo, Carlo Lovadina, Ilaria Perugia, and Giancarlo Sangalli

2003–13 *A reduced P^1–discontinuous Galerkin method*
R. Becker, E. Burman, P. Hansbo, and M.G. Larson

2003–14 *Nitsche's method combined with space–time finite elements for ALE fluid–structure interaction problems*
Peter Hansbo, Joakim Hermansson, and Thomas Svedberg

2003–15 *Stabilized Crouzeix–Raviart element for the Darcy-Stokes problem*
Erik Burman and Peter Hansbo

2003–16 *Edge stabilization for the generalized Stokes problem: a continuous interior penalty method*
Erik Burman and Peter Hansbo

2003–17 *A conservative flux for the continuous Galerkin method based on discontinuous enrichment*
Mats G. Larson and A. Jonas Niklasson

2003–18 *CAD–to–CAE integration through automated model simplification and adaptive modelling*
K.Y. Lee, M.A. Price, C.G. Armstrong, M.G. Larson, and K. Samuelsson

2003–19 *Multi-adaptive time integration*
Anders Logg

2003–20 *Adaptive computational methods for parabolic problems*
Kenneth Eriksson, Claes Johnson, and Anders Logg

2003–21 *The FEniCS project*
T. Dupont, J. Hoffman, C. Johnson, R.C. Kirby, M.G. Larson, A. Logg, and R. Scott

2003–22 *Adaptive finite element methods for LES: Computation of the mean drag coefficient in a turbulent flow around a surface mounted cube using adaptive mesh refinement*
Johan Hoffman

2003–23 *Adaptive DNS/LES: a new agenda in CFD*
Johan Hoffman and Claes Johnson

2003–24 *Multiscale convergence and reiterated homogenization of parabolic problem*
Anders Holmbom, Nils Svanstedt, and Niklas Wellander

2003–25 *On the relationship between some weak compactnesses with different numbers of scales*
Anders Holmbom, Jeanette Silfver, Nils Svanstedt, and Niklas Wellander

2003–26 *A posteriori error estimation in computational inverse scattering*
Larisa Beilina and Claes Johnson

2004–01 *Computability and adaptivity in CFD*
Johan Hoffman och Claes Johnson

2004–02 *Interpolation estimates for piecewise smooth functions in one dimension*
Anders Logg

2004–03 *Estimates of derivatives and jumps across element boundaries for multi-adaptive Galerkin solutions of ODEs*
Anders Logg

2004–04 *Multi-adaptive Galerkin methods for ODEs III: Existence and stability*
Anders Logg

2004–05 *Multi-adaptive Galerkin methods for ODEs IV: A priori error estimates*
Anders Logg

2004–06 *A stabilized non-conforming finite element method for incompressible flow*
Erik Burman and Peter Hansbo

2004–07 *On the uniqueness of weak solutions of Navier-Stokes equations: Remarks on a Clay Institute prize problem*
Johan Hoffman and Claes Johnson

2004–08 *A new approach to computational turbulence modeling*
Johan Hoffman and Claes Johnson

2004–09 *A posteriori error analysis of the boundary penalty method*
Kenneth Eriksson, Mats G. Larson, and Axel Mlqvist

2004–10 *A posteriori error analysis of stabilized finite element approximations of the helmholtz equation on unstructured grids*
Mats G. Larson and Axel Mlqvist

2004–11 *Adaptive variational multiscale methods based on a posteriori error estimation*
Mats G. Larson and Axel Mlqvist

2004–12 *Multi-adaptive Galerkin methods for ODEs V: Stiff problems*
Johan Jansson and Anders Logg

2004–13 *Algorithms for multi-adaptive time-stepping*
Johan Jansson and Anders Logg

2004–14 *Simulation of mechanical systems with individual time steps*
Johan Jansson and Anders Logg

2004–15 *Computational modeling of dynamical systems*
Johan Jansson, Claes Johnson, and Anders Logg

2004–16 *Adaptive variational multiscale methods based on a posteriori error estimation: Duality techniques for elliptic problems*
Mats G. Larson and Axel Mlqvist

2004–17 *Ultraconvergence of an interpolated finite element method for some fourth-order elliptic problems*
Andrey B. Andreev and Milena R. Racheva

2004–18 *Adaptive variational multiscale methods based on a posteriori error estimation: energy norm estimates for elliptic problems*
Mats G. Larson and Axel Mlqvist

2004–19 *Stabilized Lagrange multiplier methods for elastic contact with friction*
Per Heintz and Peter Hansbo

2005–01 *A posteriori error estimates for mixed finite element approximations of elliptic problems*
Mats G. Larson and Axel Mlqvist

2005–02 *On the numerical modeling of quasi-static crack growth in linear elastic fracture mechanics*
Per Heintz

2005–03 *Irreversibility in reversible systems I: the compressible Euler equations in 1d*
Johan Hoffman and Claes Johnson

2005–04 *Irreversibility in reversible systems II: the incompressible Euler equations*
Johan Hoffman and Claes Johnson

2005–05 *A Compiler for Variational Forms*
Robert C. Kirby and Anders Logg

2005–06 *Topological optimization of the evaluation of finite element matrices*
Robert C. Kirby, Anders Logg, L. Ridgway Scott and Andy R. Terrel

2005–07 *Modeling of resistive wall mode and its control in experiments and ITER*
Yueqiang Liu, M.S. Chu, A.M. Garofalo, Y. Gribov, M. Gryaznevich, T.C. Hender, D.F. Howell, R.J. La Haye, M. Okabayashi, S.D. Pinches, H. Reimerdes, P. de Vries, and EFDA-JET contributors

2005–08 *Continuum gradient based shape optimization of conducting shields for power frequency magnetic field mitigation*
Yueqiang Liu, P. Sousa Jr., E. Salinas, P. Cruz and J. Daalder

2005–09 *A quasi-planar incident wave excitation for time-domain scattering analysis of periodic structures*
David Ericsson, Tomas Hallerd, Brje Emilsson, Anders Bondeson and Thomas Rylander

2005–10 *Piecewise divergence free discontinuous Galerkin methods*
Peter Hansbo and Mats G. Larson