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Chalmers Finite Element Center
Chalmers University of Technology
SE–412 96 Göteborg
Sweden
Telephone: +46 (0)31 772 1000
Fax: +46 (0)31 772 3595
www.phi.chalmers.se

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MULTISCALE CONVERGENCE AND REITERATED HOMOGENIZATION OF PARABOLIC PROBLEMS

ANDERS HOLMBOM, NILS SVANSTEDT, AND NIKLAS WELLANDER

ABSTRACT. Reiterated homogenization is studied for divergence structure parabolic problems of the form $\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(a\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k}\right)\nabla u_\varepsilon\right) = f$. It is shown that under standard assumptions on the function $a\left(x, y_1, y_2, t, \tau\right)$ the sequence $\{u_\varepsilon\}$ of solutions converges weakly in $L^2(0,T;H_0^1(\Omega))$ to the solution u of the homogenized problem $\frac{\partial u}{\partial t} - \operatorname{div}\left(b(x,t)\nabla u\right) = f$.

1. INTRODUCTION

In this paper we consider the homogenization problem for the following initial-boundary value problem:

$$(1) \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(a\left(x, \dfrac{x}{\varepsilon}, \dfrac{x}{\varepsilon^2}, t, \dfrac{t}{\varepsilon^k}\right)\nabla u_\varepsilon\right) = f \text{ in } \Omega \times (0,T), \\ u_\varepsilon(x,0) = u_0(x), \\ u_\varepsilon(x,t) = 0 \text{ in } \partial\Omega \times (0,T), \end{cases}$$

where Ω is an open bounded set in $\mathbb{R}^n$, T and k are positive real numbers. Let us define $\Omega_T = \Omega \times (0,T)$ and $Y_\tau = Y_1 \times Y_2 \times (0,1)$, where $Y_1 = Y_2 = (0,1)^n$. We assume that the function $a = a(x, y_1, y_2, t, \tau)$ belongs to $C(\Omega_T; L^\infty_{per}(Y_\tau))$ and satisfies the coercivity assumption

$$\alpha|\xi|^2 \leq a\xi \cdot \xi, \ \forall \xi \in \mathbb{R}^n, \ a.e. \text{ in } \Omega_T \times Y_\tau.$$

With these structure conditions it is well-known that given $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$ there exists a unique solution $u_\varepsilon \in L^2(0,T;H_0^1(\Omega))$ to (1) with time derivative $\frac{\partial u_\varepsilon}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$ for every fixed $\varepsilon > 0$.

The homogenization problem for (1) consists in studying the asymptotic behavior of the solutions u_ε as ε tends to zero.

Homogenization problems with more than one oscillating scale is referred to as reiterated homogenization and was first introduced in [2] for linear elliptic problems. More recently the linear elliptic problem was studied in [1] and the nonlinear monotone case was treated in [6]. In the present report we prove a reiterated homogenization theorem

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Anders Holmbom, Department of Mathematics, Mid-Sweden University, SE-831 25 Östersund, Sweden, *email*: anders.holmbom@mh.se Nils Svanstedt, Department of Computational Mathematics, Chalmers University, SE-412 96 Göteborg, Sweden, *email*: nilss@math.chalmers.se Niklas Wellander, Swedish Defence Research Agency, FOI, P.O. Box 1165, SE-581 11 Linköping, Sweden. *email*: niklas@foi.se.

(Theorem 6) for the parabolic problem (1). In particular the proof of Theorem 6 will show how easy and powerful the two-scale and multi-scale convergence theory can be.

Throughout the paper we consider a sequence $\{\varepsilon_i\}$ of small positive numbers tending to zero which is denoted $\{\varepsilon\}$. Any subsequence $\{\varepsilon'\}$ of the sequence $\{\varepsilon\}$ will also be denoted $\{\varepsilon\}$.

The result of Theorem 6 is that the sequence of solutions $\{u_\varepsilon\}$ to (1) converges weakly in $L^2(0,T;H_0^1(\Omega))$ to the solution u in $L^2(0,T;H_0^1(\Omega))$ to a homogenized problem of the form

$$(2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \text{div }(b(x,t)\nabla u) = f \quad \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), \\ u(x,t) = 0 \quad \text{in } \partial\Omega \times (0,T), \end{cases}$$

where b depends on x and t but is no longer oscillating with ε. Indeed b will also depend on k, but this will be clearly spelled out in the main result (Theorem 6). As a warm up, in order to get a feeling for the interaction between the scales, we expand the solution u_ε to (1) in a multiple scales power series. Let us for the moment assume that

$$(3)\qquad u_\varepsilon(x,t) = u(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}) + \varepsilon u_1(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}) + \varepsilon^2 u_2(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}) + \dots,$$

where all the u_is are assumed to be ε-periodic in $y_1 = x/\varepsilon$, ε^2-peridic in $y_2 = x/\varepsilon^2$ and ε^k-periodic in $\tau = t/\varepsilon^k$, respectively. The chain rule transforms the differential operators as

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \frac{1}{\varepsilon^k}\frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_1} + \frac{1}{\varepsilon^2}\frac{\partial}{\partial y_2}.$$

The divergence and gradient operators transform accordingly and we denote differentiation with respect to x, y_1 and y_2 by subscripts x, y_1 and y_2, respectively. In a standard way one can now insert the series (3) into the equation (1) and identify a hierarchy of equations of siginificant orders of ε. This is performed in the Appendix in the end of the paper. In Section 1 we give some preliminaries and present some well-known and new results needed in the proof of the main result of the paper (Theorem 6) which is presented in Section 2 together with a compactness result (Theorem 5) for parabolic problems with multiple spatial scales. Section 3 is devoted to the proof of Theorem 6. It is lengthy but straightforward thanks to the preparatory Theorem 3 and Theorem 4 and the compactness Theorem 5.

2. Preliminaries

We will now recall the concept of multiscale convergence, see Allaire and Briane [1]. We will restrict ourselves to three spatial scales and two time scales as in the initial-boundary value problem (1) studied in this report.

Definition 1. *A sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ is said to multi-scale converge (with three spatial scales) to $u = u(x,y_1,y_2)$ in $L^2(\Omega \times Y_1 \times Y_2)$ if*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\varphi(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2})\,dx = \int_\Omega\int_{Y_1}\int_{Y_2} u(x,y_1,y_2)\varphi(x,y_1,y_2)\,dxdy_1dy_2,$$

for all functions $\varphi \in L^2(\Omega; C_{per}(Y_1 \times Y_2))$.

Allaire and Briane proved the following compactness results:

Theorem 1. *Let* $\{u_\varepsilon\}$ *be bounded sequence in* $L^2(\Omega)$. *Then there exists a subsequence, still denoted* $\{u_\varepsilon\}$, *and a function* $u = u(x, y_1, y_2)$ *in* $L^2(\Omega \times Y_1 \times Y_2)$ *such that* u_ε *multi-scale converges to* u.

Theorem 2. *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $H^1(\Omega)$. *Then there exist subsequences*

$$u_\varepsilon \to u \text{ strongly in } L^2(\Omega),$$

and

$$\nabla u_\varepsilon \to \nabla_x u(x) + \nabla_{y_1} u_1(x, y_1) + \nabla_{y_2} u_2(x, y_1, y_2),$$

in the multi-scale sense, where $u \in H^1(\Omega)$, $u_1 \in L^2(\Omega;\ H^1_{per}(Y_1))$ *and* $u_2 \in L^2(\Omega \times Y_1;\ H^1_{per}(Y_2))$

We can also consider bounded functions in L^2 depending on the time variable t.

Definition 2. *A sequence* $\{u_\varepsilon\}$ *in* $L^2(\Omega \times (0, T))$ *is said to multi-scale converge in space-time with three spatial and two temporal scales if, for a constant* $k > 0$,

$$\lim_{\varepsilon \to 0} \int_\Omega \int_0^T u_\varepsilon(x, t) \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k})\, dxdt =$$

$$\int_\Omega \int_0^T \int_{Y_1} \int_{Y_2} \int_0^1 u(x, y_1, y_2, t, \tau) \varphi(x, y_1, y_2, t, \tau)\, dxdy_1dy_2dtd\tau.$$

where $u \in L^2(\Omega \times (0, T) \times Y_1 \times Y_2 \times (0, 1))$ *for all* $\varphi \in L^2(\Omega \times (0, T); C_{per}(Y_1 \times Y_2 \times (0, 1)))$.

We also have:

Proposition 1. *(Space) Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $L^2(0, T; H^1(\Omega))$ *such that its distributional temporal derivative* $\{u'_\varepsilon\}$ *is a bounded sequence in* $L^2(0, T; (H^1(\Omega))')$. *Then its well-known that* $\{u_\varepsilon\}$ *is compact in* $L^2((0, T) \times \Omega)$ *and there exist subsequences*

$$u_\varepsilon \to u \text{ strongly in } L^2((0, T) \times \Omega),$$

and

$$\nabla u_\varepsilon \to \nabla_x u(x, t) + \nabla_{y_1} u_1(x, t, y_1) + \nabla_{y_2} u_2(x, t, y_1, y_2),$$

in the multi-scale sense, where $u \in L^2(0, T; H^1(\Omega))$, $u_1 \in L^2((0, T) \times \Omega;\ H^1_{per}(Y_1))$ *and* $u_2 \in L^2((0, T) \times \Omega \times Y_1;\ H^1_{per}(Y_2))$.

Corollary 1. *(Space-time) Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $L^2(0, T; H^1(\Omega))$ *such that its distributional derivative* $\{u'_\varepsilon\}$ *is a uniformly bounded sequence in* $L^2(0, T; (H^1(\Omega))')$. *Then there exist subsequences*

$$u_\varepsilon \to u \text{ strongly in } L^2((0, T) \times \Omega),$$

and

$$\nabla u_\varepsilon \to \nabla_x u(x, t) + \nabla_{y_1} u_1(x, t, y_1, \tau) + \nabla_{y_2} u_2(x, t, y_1, y_2, \tau),$$

in the multi-scale sense in space-time, where $u \in L^2((0, T); H^1(\Omega))$, $u_1 \in L^2((0, T) \times \Omega \times (0, 1);\ H^1_{per}(Y_1))$ *and* $u_2 \in L^2((0, T) \times \Omega \times Y_1 \times (0, 1);\ H^1_{per}(Y_2))$.

In this work we will not use the following observation.

Remark 1. *If $\{u_\varepsilon\}$ is bounded in $H^1(0,T;H^1(\Omega))$, then the time derivative splits. By using test functions oscillating in time with frequency ε, i.e. $\varphi(x,t,\frac{t}{\varepsilon})$ the split yields the existence of a local function u_1 such that*

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u}{\partial t} + \frac{\partial u_1}{\partial \tau},$$

in the multi-scale sense (in time), where $u \in H^1((0,T)\times\Omega)$ and where $u_1 \in L^2((0,T);H^1_{per}(0,1)\times H^1(\Omega))$. If we instead use test functions oscillating in time with frequency ε^2, i.e. $\varphi(x,t,\frac{t}{\varepsilon^2})$, then the split yields another local function u_2, i.e.,

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u}{\partial t} + \frac{\partial u_2}{\partial \tau}$$

in the multi-scale sense, where $u \in H^1((0,T)\times\Omega)$ and where $u_2 \in L^2((0,T);H^1_{per}(0,1)\times H^1(\Omega))$.

Remark 2. *The split of the time derivative is discussed in [7] and is proved analogously to the gradient split. In the main Theorem 6 we do not have H^1-a priori bounds on the time derivative so therefore there occur no split in the time derivative. But as seen in the Appendix, a formal expansion yields time split derivatives in the $k=1$ and $k=2$ cases, respectively. However, that is only formally and is never used since the local derivatives vanish when the equations are averaged over fast time.*

We continue by stating and proving two theorems that will be crucial in the proof of the main Theorem 6. A similar result is earlier proved in Holmbom [5].

Theorem 3. *Let $\{u_\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$ and let u and u_1 be defined as in Theorem 2. Then,*

$$\lim_{\varepsilon\to 0}\int_\Omega \left(\frac{u_\varepsilon(x)-u(x)}{\varepsilon}\right)\varphi\left(x,\frac{x}{\varepsilon}\right)\,dx = \int_\Omega\int_{Y_1} u_1(x,y_1)\varphi(x,y_1)\,dy_1dx,$$

for all $\varphi(x,y_1)=\varphi_1(x)\varphi_2(y_1)$ where $\varphi_1\in C_0^\infty(\Omega)$ and $\varphi_2\in C^\infty_{per}(Y_1)$ with mean value zero over Y_1.

Proof: From Theorem 2, by choosing test functions $\psi(x,y)=\psi_1(x)\psi_2(y)$ in $C_0^\infty(\Omega;C^\infty_{per}(Y_1;\mathbb{R}^n))$, $\psi_1\in C_0^\infty(\Omega)$, $\psi_2\in C^\infty_{per}(Y_1;\mathbb{R}^n)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega (\nabla u_\varepsilon(x)-\nabla u(x))\cdot\psi_1(x)\psi_2(\frac{x}{\varepsilon})dx = \int_\Omega\int_{Y_1}\nabla u_1(x,y_1)\cdot\psi_1(x)\psi_2(y_1)dy_1dx.$$

The divergence theorem applied on both sides gives

$$\lim_{\varepsilon\to 0}\int_\Omega (u_\varepsilon(x)-u(x))\left(\psi_2(\frac{x}{\varepsilon})\mathrm{div}_x\psi_1(x)+\psi_1(x)\varepsilon^{-1}\mathrm{div}_{y_1}\psi_2(\frac{x}{\varepsilon})\right)dx =$$
$$\int_\Omega\int_{Y_1} u_1(x,y_1)\psi_1(x)\mathrm{div}_{y_1}\psi_2(y_1)dy_1dx.$$

By the mean value zero condition over Y_1 for φ_2 we can apply the well-known Fredholm alternative and conclude that there exists a unique Y_1-periodic solution $\eta\in C^\infty_{per}(Y_1)$ to

$$\begin{cases} \mathrm{div}_{y_1}(\nabla_{y_1}\eta)=\varphi_2, & \text{in } Y_1 \\ \eta\in C^\infty_{per}(Y_1;\mathbb{R}^n)). \end{cases}$$

Now we simply let $\varphi_1 = \psi_1$ and $\psi_2 = \nabla_{y_1}\eta$ to obtain $\varphi_2 = \mathrm{div}_{y_1}\psi_2$. The strong convergence of $\{u_\varepsilon\}$ in $L^2(\Omega)$ to u in Theorem 2 gives the result. □

As a consequence of Theorem 2 we can extend the result of Theorem 3 to the case of 3 scales and state the following:

Theorem 4. *Assume that $u_1(x,y)$ is of Caratheodory type and let $\{u_\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$. Further let u, u_1, u_2 be defined by the limit in Theorem 2 . Then*

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{u_\varepsilon(x) - u(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)}{\varepsilon^2}\varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)\, dx =$$

$$\int_\Omega\int_{Y_1}\int_{Y_2} u_2(x,y_1,y_2)\varphi(x,y_1,y_2)\, dx dy_1 dy_2.$$

in $L^2(\Omega\times Y_1; H^1_{per}(Y_2))$ for $\varphi(x,y_1,y_2) = \varphi_1(x)\varphi_2(y_1)\varphi_3(y_2)$ where $\varphi_1 \in C_0^\infty(\Omega)$ and $\varphi_2, \varphi_3 \in C^\infty_{per}(Y)$ with mean value zero over Y.

Remark 3. *An example of a function which has the regularity conditions which allows a scaling of the function $u_1 = u_1(x,y_1)$ is given in Cioranescu and Donato* [3]*, Chapter 9. Suppose*

$$u_1(x,y_1) = \sum_{j=1}^n \omega_j(y_1)\frac{\partial u_0}{\partial x_j}(x)$$

where $\nabla_{y_1}\omega_i \in L^r(Y_1;\mathbb{R}^n)$, $i = 1, \ldots, n$ and $\nabla_x u \in L^s(\Omega;\mathbb{R}^n)$, with $1 \le r, s < \infty$ and $1/r + 1/s = 1/2$. Then, for test functions $\varphi \in C_0^\infty(\Omega; C^\infty_{per}(Y_1;\mathbb{R}^n))$,

$$\int_\Omega \nabla_{y_1} u_1(x, \frac{x}{\varepsilon})\cdot\varphi(x,\frac{x}{\varepsilon})dx \to \int_\Omega\int_{Y_1}\nabla_{y_1}u_1(x,y_1)\cdot\varphi(x,y)dy_1dx.$$

Remark 4. *The result remains valid also for the case $r = s = 2$, but then the two-scale convergence takes place in L^1.*

Proof: Let us choose test functions $\psi \in C_0^\infty(\Omega; C^\infty_{per}(Y_1\times Y_2 : \mathbb{R}^n))$ The result of Theorem 2 says that

$$\lim_{\varepsilon\to 0}\int_\Omega\left(\nabla_x u_\varepsilon(x) - \nabla_x u(x) - \nabla_{y_1}u_1\left(x,\frac{x}{\varepsilon}\right)\right)\cdot\psi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\, dx =$$

$$\int_\Omega\int_{Y_1}\int_{Y_2}\nabla_{y_2}u_2(x,y_1,y_2)\cdot\psi(x,y_1,y_2)\, dx dy_1 dy_2.$$

An integration by parts on both sides gives

$$\lim_{\varepsilon\to 0}\int_\Omega\left(u_\varepsilon(x) - u(x) - \varepsilon u_1\left(x,\frac{x}{\varepsilon}\right)\right)\left(\left(\mathrm{div}_x + \varepsilon^{-1}\mathrm{div}_{y_1} + \varepsilon^{-2}\mathrm{div}_{y_2}\right)\psi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)\, dx =$$

$$\int_\Omega\int_{Y_1}\int_{Y_2} u_2(x,y_1,y_2)\mathrm{div}_{y_2}\psi(x,y_1,y_2)\, dx dy_1 dy_2.$$

By Theorem 3

$$\lim_{\varepsilon\to 0}\int_\Omega\left(u_\varepsilon(x) - u(x) - \varepsilon u_1\left(x,\frac{x}{\varepsilon}\right)\right)\left(\left(\mathrm{div}_x + \varepsilon^{-1}\mathrm{div}_{y_1}\right)\psi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)\, dx = 0.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{u_\varepsilon(x)-u(x)-\varepsilon u_1\left(x,\frac{x}{\varepsilon}\right)}{\varepsilon^2}\mathrm{div}_{y_2}\psi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\,dx =$$
$$\int_\Omega\int_{Y_1}\int_{Y_2} u_2(x,y_1,y_2)\mathrm{div}_{y_2}\psi(x,y_1,y_2)\,dxdy_1dy_2.$$

Referring to Lemma 2.4 in [8] we can argue as in Theorem 3 and obtain any φ as $\varphi = \mathrm{div}_{y_2}\psi$. □

Remark 5. *If $\nabla_y u_1 \in L^r(Y;\mathbb{R}^n)$ and $\nabla_x u \in L^s(\Omega;\mathbb{R}^n)$, where $1\le r,\ s<\infty$, $1/r+1/s=1/2$, then the convergence in Theorem 4 takes place in L^2. However, since the limit u_2 is an element in $L^2(\Omega\times Y_1;\ H^1_{per}(Y_2))$, this is just a technical argument.*

3. The main results

Let us rewrite (1) in the variational formulation:
Find $u_\varepsilon \in L^2(0,T;H^1_0(\Omega))$ such that

$$-\int_{\Omega_T} u_\varepsilon(x,t)\frac{\partial\varphi(x,t)}{\partial t}\,dxdt + \int_{\Omega_T} a\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}\right)\nabla u_\varepsilon(x,t)\cdot\nabla\varphi(x,t)\,dxdt$$

$$= \int_{\Omega_T} f(x,t)\varphi(x,t)\,dxdt \quad \text{for all} \quad \varphi\in L^2(0,T;H^1_0(\Omega)) \tag{4}$$

$$u_\varepsilon(x,0)=u_0(x).$$

We first observe that by the structure conditions on $a(x,y_1,y_2,t,\tau)$ one immediately obtains the following a priori estimates (see e.g. [3] Ch. 11):

$$\|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))}\le C \quad \|\frac{\partial u_\varepsilon}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))}\le C$$

We begin with the following compactness result:

Theorem 5. *Let $\{u_\varepsilon\}$ be a sequence of solutions to the variational problem (4) above. By sending $\varepsilon\to 0$ the sequence $\{u_\varepsilon\}$ multi-scale converges to the unique solution u to the following problem*

$$-\int_{\Omega_T} u(x,t)\frac{\partial\varphi(x,t)}{\partial t}\,dxdt$$

$$+\int_{\Omega_T}\left[\int_{Y_\tau} a(x,y_1,y_2,t,\tau)[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]dy_1dy_2\,d\tau\right]\cdot\nabla\varphi(x,t)\,dxdt \tag{5}$$

$$= \int_{\Omega_T} f(x,t)\varphi(x,t)\,dxdt \quad \text{for all} \quad \varphi\in L^2(0,T;H^1_0(\Omega)),$$

with initial condition

$$u_\varepsilon(x,0)=u_0(x),$$

where

$$u=u(x,t),\quad u_1=u_1(x,y_1,t,\tau),\quad \text{and}\quad u_2=u_2(x,y_1,y_2,t,\tau). \tag{6}$$

Proof: By the structure conditions on the function a, the product function

$$a\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k}\right) \varphi(x,t)$$

is an admissible test function and therefore by using Corollary 1 and by exploiting the strong $L^2((0,T)\times\Omega)$- convergence of u_ε, which is an immediate consequence of the a priori estimates above, the result follows. □

Remark 6. *It now remains to find the equations that u_1 and u_2 satisfy. The dynamics will be captured by considering test functions which capture the oscillations in time. Due to the spatial and temporal oscillations in the coefficient we seek for u_ε of the form (3). But, due to Theorem 5 we know that the candidates are of the form (6). This can also be verified, at least formally, by equating the different orders in ε of (1) while inserting (3). In the Appendix we perform a multiple scales expansion in order to derive the correct equations for u_1 and u_2 respectively.*

Our aim is now to derive rigorously the local equation. We will use Theorem 3, Theorem 4 and Theorem 5 together with test functions which are in resonance with the oscillating coefficients $a_\varepsilon = a(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k})$. Before stating and proving the reiterated homogenization theorem we introduce some notations and abreviations: We simply write a to denote $a(x, y_1, y_2, t, \tau)$ and u, u_1 and u_2 to denote $u(x,t)$, $u_1(x, y_1, t, \tau)$ and $u_2(x, y_1, y_2, t, \tau)$, respectively. We also write $dy_\tau dx_T$ to denote $dy_1 dy_2 dx d\tau dt$. Moreover, we denote by φ_ε smooth oscillating test functions of the types $\varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k})$, $\varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t)$, $\varphi(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^k})$ or $\varphi(x, \frac{x}{\varepsilon}, t)$.

Theorem 6. *(Reiterated homogenization) Consider the sequence of initial-boundary value problems (1),*

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left(a\left(x, \dfrac{x}{\varepsilon}, \dfrac{x}{\varepsilon^2}, t, \dfrac{t}{\varepsilon^k}\right)\nabla u_\varepsilon\right) = f \text{ in } \Omega\times(0,T), \\ u_\varepsilon(x,0) = u_0(x), \\ u_\varepsilon(x,t) = 0 \text{ in } \partial\Omega\times(0,T), \end{cases}$$

It follows that as $\varepsilon \to 0$:

$$u_\varepsilon \to u, \quad in \quad L^2(0,T; H_0^1(\Omega)) \quad weakly,$$

$$a\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k}\right)\nabla u_\varepsilon \to b(x,t)\nabla u, \quad in \quad L^2(\Omega)^n \quad weakly,$$

where b is the homogenized coefficient and where u solves the homogenized problem (2):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\,(b(x,t)\nabla u) = f \quad in\ \Omega\times(0,T), \\ u(x,0) = u_0(x), \\ u(x,t) = 0 \quad in\ \partial\Omega\times(0,T), \end{cases}$$

or in variational form (5):

$$-\int_{\Omega_T} u(x,t)\frac{\partial\varphi(x,t)}{\partial t}\,dxdt$$

$$+\int_{\Omega_T}\left[\int_{Y_\tau} a(x,y_1,y_2,t,\tau)[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]dy_1dy_2\,d\tau\right]\cdot\nabla\varphi(x,t)\,dxdt$$

$$=\int_{\Omega_T} f(x,t)\varphi(x,t)\,dxdt$$

The unknown functions u, u_1 and u_2 together satisfy a characteristic system of local equations of different order of ε. Depending on the value of the oscillation power k in the fast time variable, there are 7 different cases of systems of local equations namely:
$0<k<2$

$$\begin{cases}\displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,dy_\tau dx_T=0.\\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_1}\varphi(x,y_1,t,\tau)\,dy_\tau dx_T=0.\end{cases}$$

k=2

$$\begin{cases}\displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,dy_\tau dx_T=0.\\ \displaystyle-\int_{\Omega_T}\int_{Y_\tau} u_1(x,y_1,t,\tau)\frac{\partial\varphi}{\partial\tau}(x,y_1,t,\tau)\,dy_\tau dx_T+\\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_1}\varphi(x,y_1,t,\tau)\,dy_\tau dx_T=0.\end{cases}$$

$2<k<3$

$$\begin{cases}\displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,t,\tau)\,dy_\tau dx_T=0.\\ \displaystyle-\int_{\Omega_T}\int_{Y_\tau} u_1\frac{\partial\varphi}{\partial\tau}(x,y_1,t,\tau)\,dy_\tau dx_T=0.\\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_1}\varphi(x,y_1,t,\tau)\,dy_\tau dx_T=0.\end{cases}$$

k = 3

$$\begin{cases}\displaystyle-\int_{\Omega_T}\int_{Y_\tau} u_1\frac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,dy_\tau dx_T\\ \displaystyle+\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,dy_\tau dx_T=0\\ \displaystyle-\int_{\Omega_T}\int_{Y_\tau} u_2\frac{\partial\varphi}{\partial\tau}(x,y_1,t,\tau)\,dy_\tau dx_T+\\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_1}\varphi(x,y_1,t,\tau)\,dy_\tau dx_T=0.\end{cases}$$

$3 < k < 4$

$$\begin{cases} -\int_{\Omega_T}\int_{Y_\tau} u_1 \dfrac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ \int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ -\int_{\Omega_T}\int_{Y_\tau} u_2 \dfrac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ \int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]dy_2 d\tau\right]\cdot\nabla_{y_1}\varphi(x,y_1,t)\,dy_1 dx dt = 0. \end{cases}$$

$k = 4$

$$\begin{cases} -\int_{\Omega_T}\int_{Y_\tau} u_1 \dfrac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ -\int_{\Omega_T}\int_{Y_\tau} u_2 \dfrac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,dy_\tau dx_T + \\ \int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ \int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]dy_2 d\tau\right]\cdot\nabla_{y_1}\varphi(x,y_1,t)\,dy_1 dx dt = 0. \end{cases}$$

$k > 4$

$$\begin{cases} -\int_{\Omega_T}\int_{Y_\tau} u_1 \dfrac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ -\int_{\Omega_T}\int_{Y_\tau} u_2 \dfrac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ \int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0. \\ \int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]dy_2 d\tau\right]\cdot\nabla_{y_1}\varphi(x,y_1,t)\,dy_1 dx dt = 0. \end{cases}$$

Remark 7. *The homogenized map b is derived in the usual way by a separation of variables. Lets consider the variational form of the ε^{-2}-equation for the case $0 < k < 2$:*

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,dy_\tau dx_T = 0$$

By linearity we can decouple variables:

$$u_2(x,t,y_1,y_2,\tau) = -\left(\nabla_x u(x,t) + \nabla_{y_1} u_1(x,t,y_1,\tau)\right)\cdot w_2(y_2,\tau).$$

We can now write the decoupled local ε^{-2}-equation as the parameter dependent (parameter τ) problem:
Find $w_2^k(\cdot,\tau) \in H^1_{per}(Y_2)$ such that for almost every $\tau \in (0,1)$

$$\int_{Y_2} a_{ij}(x,t,y_1,y_2,\tau)\left(\delta_{jk} - \frac{\partial w_2^k(y_2,\tau)}{\partial_{y_{2_j}}}\right)\frac{\partial \varphi(y_2)}{\partial_{y_{2_i}}} dy_2 = 0,$$

for all $\varphi \in C^\infty_{per}(Y_2)$, and we define

$$b^1_{ik}(x,t,y_1,\tau) = \int_{Y_2} a_{ij}(x,t,y_1,y_2,\tau)\left(\delta_{jk} - \frac{\partial w_2^k(y_2,\tau)}{\partial_{y_{2_j}}}\right) dy_2.$$

The local decoupled ε^{-1}-equation can then be written (using the same traditional arguments as above):
Find $v_1^k(\cdot,\tau) \in H^1_{per}(Y_1)$, such that for almost every $\tau \in (0,1)$

$$\int_{Y_1} b^1_{ij}(x,t,y_1,y_2,\tau)\left(\delta_{jk} - \frac{\partial v_1^k(y_1,\tau)}{\partial_{y_{1_j}}}\right)\frac{\partial \varphi(y_1)}{\partial_{y_{1_i}}} dy_1 = 0,$$

for all $\varphi \in C^\infty_{per}(Y_1)$.
Finally we define

$$b_{ik}(x,t) = \int_{Y_1}\int_0^1 b^1_{ij}(x,t,y_1,\tau)\left(\delta_{jk} - \frac{\partial v_1^k(y_1,\tau)}{\partial_{y_{1_j}}}\right) dy_1 d\tau.$$

This procedure is standard and analogous for the different cases. The existence and uniqueness of local solutions is carried out in [4] *in the linear periodic case.*

4. Proof of Theorem 6

Let us choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k})$. By the chain rule the variational formulation of (1) reads: Find $u_\varepsilon \in L^2(0,T; H^1_0(\Omega))$ such that

$$-\int_{\Omega_T} u_\varepsilon\left(\frac{\partial \varphi_\varepsilon}{\partial t} + \frac{\varepsilon^{-k}\partial \varphi_\varepsilon}{\partial \tau}\right) dxdt + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt$$

$$= \int_{\Omega_T} f\varphi_\varepsilon \, dxdt \quad \forall \varphi_\varepsilon \in L^2(0,T : H^1_0(\Omega)) \tag{7}$$

$$u_\varepsilon(x,0) = u_0(x).$$

Let us now case by case show that the local equations for u, u_1 and u_2 will appear as multiscale limits of (7) with appropriate choices of test functions φ_ε. As the formal analysis in the Appendix shows, there are seven different significant cases for k to be considered: $0 < k < 2$, $k = 2$, $2 < k < 3$, $k = 3$, $3 < k < 4$, $k = 4$ and $k > 4$, respectively.

$0 < k < 2$

Step 1: Let us consider (7). We choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k})$. A multiplication by ε^2 on both sides of the equation and a limit passage yields the ~ -2 equation:

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, dy_\tau dx_T, = 0.$$

Step 2: Choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^k})$. and study the equation

$$-\int_{\Omega_T} u_\varepsilon \left(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-k} \frac{\partial \varphi_\varepsilon}{\partial \tau} \right) dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\, dxdt = \int_{\Omega_T} f\varphi_\varepsilon\, dxdt$$

A multiplication by ε on both sides of the equation and a limit passage yields the ~ -1 equation:

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, dy_\tau dx_T = 0.$$

k=2

Step 1: We choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^2})$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon \left(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-2} \frac{\partial \varphi_\varepsilon}{\partial \tau} \right) dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, dxdt = \int_{\Omega_T} f\varphi_\varepsilon\, dxdt \tag{8}$$

A multiplication by ε^2 on both sides of the equation and a limit passage in (8), using Corollary 1, yields the ~ -2 equation:

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

Step 2: We chose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^2})$ and consider the difference between (7) and the weak limit (5) in Theorem 5.

$$-\int_{\Omega_T} (u_\varepsilon - u) \left(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-2} \frac{\partial \varphi_\varepsilon}{\partial \tau} \right) dxdt +$$

$$\int_{\Omega_T} \left(a_\varepsilon \nabla u_\varepsilon - \int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\, dyd\tau \right) \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\, dxdt = 0.$$

A multiplication by ε^1 on both sides of the equation and a limit passage, where Theorem 3 is used in the first term yields the ~ -1 equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_1(x, y_1, t, \tau) \frac{\partial \varphi}{\partial \tau}(x, y_1, t, \tau)\, dy_\tau dx_T +$$

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, dy_\tau dx_T = 0.$$

$2 < k < 3$

Step 1: Choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, dxdt = \int_{\Omega_T} f\varphi_\varepsilon\, dxdt.$$

Multiplication by ε^2 and a limit passage yields the ~ -2 equation:

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, t, \tau)\, dy_\tau dx_T = 0.$$

Step 2: Choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^2})$ and consider

$$-\int_{\Omega_T} \left[(u_\varepsilon - u)\left(\frac{\partial \varphi_\varepsilon}{\partial t} + \frac{\varepsilon^{-k}\partial \varphi_\varepsilon}{\partial \tau}\right) - u\frac{\partial \varphi_\varepsilon}{\partial t}\right] dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, dxdt = \int_{\Omega_T} f\varphi_\varepsilon\, dxdt.$$

A multiplication by ε^{k-1} and a limit passage, where Theorem 3 is used yields the $\sim -k+1$ equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial \varphi}{\partial \tau}(x, y_1, t, \tau)\, dy_\tau dx_T = 0.$$

And hence $u_1 = u_1(x, y_1, t)$.

Step 3: Choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\, dxdt = \int_{\Omega_T} f\varphi_\varepsilon\, dxdt.$$

Multiplication by ε and a limit passage yields the ~ -1 equation:

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, dy_\tau dx_T = 0.$$

k=3

Step 1: We consider again the difference between (7) and the weak limit (5) in Theorem 5, i.e.

$$-\int_{\Omega_T} (u_\varepsilon - u)\left(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-3}\frac{\partial \varphi_\varepsilon}{\partial \tau}\right) dxdt +$$

$$\int_{\Omega_T}\left(a_\varepsilon \nabla u_\varepsilon - \int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\, dy d\tau\right)\cdot(\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, dxdt = 0$$

A multiplication by ε^2 on both sides of the equation and a limit passage, where Theorem 3 is used, yields the ~ -2 equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial\varphi}{\partial\tau}(x, y_1, y_2, t, \tau)\, dy_\tau dx_T$$

$$+\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\cdot\nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0. \tag{9}$$

Step 2: We chose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^3})$. Scale $y_1 = x/\varepsilon$ in u_1, multiply (9) by ε and subtract this from the difference between (7) and (5). This gives

$$-\int_{\Omega_T}(u_\varepsilon - u - \varepsilon u_1)\left(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-3}\frac{\partial\varphi_\varepsilon}{\partial\tau}\right) dxdt - \int_{\Omega_T}\varepsilon u_1 \frac{\partial\varphi_\varepsilon}{\partial t} dxdt +$$

$$\int_{\Omega_T}\left(a_\varepsilon\nabla u_\varepsilon - \int_{Y_\tau} a[\cdot]\, dy d\tau - \varepsilon\int_{Y_2} a[\cdot]\, dy_2\right)\cdot(\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\, dxdt = \int_{\Omega_T} f\varphi_\varepsilon\, dxdt,$$

where $a[\cdot] = a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] = a(x, y_1, y_2, t, \tau)[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]$. A multiplication by ε^1 on both sides of the equation and a limit passage, where Theorem 4 is used, yields the ~ -1 equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial\varphi}{\partial\tau}(x, y_1, t, \tau)\, dy_\tau dx_T +$$

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\cdot\nabla_{y_1}\varphi(x, y_1, t, \tau)\, dy_\tau dx_T = 0.$$

$3 < k < 4$

Step 1: We still choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k})$. But now we study

$$-\int_{\Omega_T}(u_\varepsilon - u)\left(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-k}\frac{\partial\varphi_\varepsilon}{\partial\tau}\right) - u\frac{\partial\varphi_\varepsilon}{\partial t}\, dxdt + \int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon\cdot(\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, dxdt$$

$$= \int_{\Omega_T} f\varphi_\varepsilon\, dxdt$$

A multiplication by ε^{k-1} on both sides of the equation and a limit passage, where Theorem 3 is used, yields the $\sim -k+1$ equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial\varphi}{\partial\tau}(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

From this we conclude that $u_1 = u_1(x, y_1, t)$, i.e. independent of τ.

Step 2: Choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial\varphi_\varepsilon}{\partial t}\, dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt.$$

A multiplication by ε^2 and a limit passage yields the ~ -2 equation:

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

Step 3: Choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon^k})$, scale $y_1 = x/\varepsilon$ in u_1 and study

$$-\int_{\Omega_T} (u_\varepsilon - u - \varepsilon u_1)\left(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial\varphi_\varepsilon}{\partial\tau}\right) - (u + \varepsilon u_1)\frac{\partial\varphi_\varepsilon}{\partial t}\, dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^{k-2} on both sides of the equation and a limit passage, where Theorem 4 is used, yields the $\sim -k+2$ equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial\varphi}{\partial\tau}(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

And hence $u_2 = u_2(x, y_1, y_2, t)$, i.e. independent of τ.

Step 4: Next we choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial\varphi_\varepsilon}{\partial t}\, dxdt + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^1 on both sides of the equation and a limit passage yields the ~ -1 equation:

$$\int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2] dy_2 d\tau\right] \cdot \nabla_{y_1}\varphi(x, y_1, t)\, dy_1 dxdt = 0.$$

k=4

Step 1: We choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^4})$. Then we study

$$-\int_{\Omega_T} (u_\varepsilon - u)\left(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial\varphi_\varepsilon}{\partial\tau}\right) - u\frac{\partial\varphi_\varepsilon}{\partial t}\, dxdt + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt$$

$$= \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^3 on both sides of the equation and a limit passage, where Theorem 3 is used, yields the ~ -3 equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial\varphi}{\partial\tau}(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

From this we conclude that $u_1 = u_1(x, y_1, t)$, i.e. independent of τ.

Step 2: Scale $y_1 = x/\varepsilon$ in u_1 and study

$$-\int_{\Omega_T} (u_\varepsilon - u - \varepsilon u_1)\left(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial \varphi_\varepsilon}{\partial \tau}\right) - (u + \varepsilon u_1)\frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^2 on both sides of the equation and a limit passage, where Theorem 4 is used, yields the ~ -2 equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \,\frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\, dy_\tau dx_T +$$

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

Step 3: Next we choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t)$. This yields

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^1 on both sides of the equation and a limit passage yields the ~ -1 equation:

$$\int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] dy_2 d\tau\right] \cdot \nabla_{y_1}\varphi(x, y_1, t)\, dy_1 dxdt = 0.$$

$k > 4$

Step 1: We choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k})$. And we study

$$-\int_{\Omega_T} (u_\varepsilon - u)\left(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial \varphi_\varepsilon}{\partial \tau}\right) - u\frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt$$

$$= \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^{k-1} on both sides of the equation and a limit passage, where Theorem 3 is used, yields the $\sim -k+1$ equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \,\frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

From this we conclude that $u_1 = u_1(x, y_1, t)$, i.e. independent of τ.

Step 2: Scale $y_1 = x/\varepsilon$ in u_1 and study

$$-\int_{\Omega_T} (u_\varepsilon - u - \varepsilon u_1)\left(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-k}\frac{\partial \varphi_\varepsilon}{\partial \tau}\right) - (u + \varepsilon u_1)\frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt +$$

$$\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^{k-2} on both sides of the equation and a limit passage, where Theorem 4 is used, yields the $\sim -k+2$ equation:

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

From this we conclude that $u_2 = u_2(x, y_1, y_2, t)$, i.e. independent of τ.

Step 3: Next we choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^2 on both sides and a limit passage yields the ~ -2 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, dy_\tau dx_T = 0.$$

Step 4: Next we choose test functions $\varphi_\varepsilon = \varphi(x, \frac{x}{\varepsilon}, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\, dxdt + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon \, dxdt = \int_{\Omega_T} f\varphi_\varepsilon \, dxdt$$

A multiplication by ε^1 on both sides of the equation and a limit passage yields the ~ -1 equation:

$$\int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] dy_2 d\tau\right] \cdot \nabla_{y_1}\varphi(x, y_1, t)\, dy_1 dxdt = 0.$$

□

Remark 8. *Theorem 7 easily generalizes to the case of N spatial scales and more than one temporal scale. The difference is that the number of intervals to be studied increases. Also one needs to prove a generalization of Theorem 4 to the case of N scales.*

Remark 9. *In the present paper we have analyzed a prototype problem in order to understand analytically the mechanism when more fine scales are added to the problem. We see that the occurence of phenomena like resonances increases and we can obtain a variety of local effects which in the end has a large impact on the global behaviour of the solution. Especially we note that by adding spatial scales the problem becomes more and more sensitive to a perturbation with respect to the number k.*

5. Appendix: Multiple Scales Expansions

Let us revisit the expansion (3). By the chain rule we have

$$\frac{\partial u_\varepsilon}{\partial t} = \left(\frac{\partial}{\partial t} + \varepsilon^{-k}\frac{\partial}{\partial \tau}\right)(u + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)$$

and

$$-\mathrm{div}(a\nabla u_\varepsilon) = -(\mathrm{div}_x + \varepsilon^{-1}\mathrm{div}_{y_1} + \varepsilon^{-2}\mathrm{div}_{y_2})\left[a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) + \right.$$

$$+ \varepsilon(\nabla_x u_1 + \nabla_{y_1} u_2 + \nabla_{y_2} u_3) + \ldots)] \,.$$

The three relevant orders of ε to study are -2, -1 and 0. We will use below the fact that we can not verify the existence of the terms $\nabla_x u_1, \nabla_{y_1} u_2$ and $\nabla_{y_2} u_3$ in L^2 by the multiscale compactness Theorem 5. We therefore omit their contribution also in the formal expansion. With higher regularity they might exist and this will lead to a more complex array of local problems. We just point out in the cases $k = 1$ and $k = 2$ that there occur, formally, two time derivatives in the zero order equation. However, the local time derivative vanish after an averaging in local time. Compare with Remark 1 where this is explained and with Remark 2 above. The structure of the hierarchy of equations will depend on $k > 0$. It turns out that there are 7 different significant cases to consider, namely: $0 < k < 2$, $k = 2$, $2 < k < 3$, $k = 3$, $3 < k < 4$, $k = 4$ and $k > 4$, respectively. We choose $k = 1$ for the case $0 < k < 2$ in order to point out the remark from above.

$k = 1$:

$$\sim -2\colon \quad -\mathrm{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$$

$$\sim -1\colon \ \frac{\partial u}{\partial \tau} - \mathrm{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$$

$$\sim 0\colon \quad \frac{\partial u}{\partial t} + \frac{\partial u_1}{\partial \tau} - \mathrm{div}_x(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$$

$k = 2$:

$$\sim -2\colon \quad \frac{\partial u}{\partial \tau} - \mathrm{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$$

$$\sim -1\colon \quad \frac{\partial u_1}{\partial \tau} - \mathrm{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$$

$$\sim 0\colon \quad \frac{\partial u}{\partial t} + \frac{\partial u_2}{\partial \tau} - \mathrm{div}_x(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$$

$2 < k < 3$:

$$\sim -k\colon \quad \frac{\partial u}{\partial \tau} = 0.$$

$$\sim -2\colon \quad -\mathrm{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$$

$$\sim -k+1\colon \quad \frac{\partial u_1}{\partial \tau} = 0.$$

$$\sim -1\colon \quad -\mathrm{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$$

$$\sim 0\colon \quad \frac{\partial u}{\partial t} - \mathrm{div}_x(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$$

$k = 3$:

~ -3: $\dfrac{\partial u}{\partial \tau} = 0.$

~ -2: $\dfrac{\partial u_1}{\partial \tau} - \mathrm{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

~ -1: $\dfrac{\partial u_2}{\partial \tau} - \mathrm{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

~ 0: $\dfrac{\partial u}{\partial t} - \mathrm{div}_x(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$

$3 < k < 4$:

$\sim -k$: $\dfrac{\partial u}{\partial \tau} = 0.$

$\sim -k+1$: $\dfrac{\partial u_1}{\partial \tau} = 0.$

~ -2: $-\mathrm{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

$\sim -k+2$: $\dfrac{\partial u_2}{\partial \tau} = 0.$

~ -1: $-\mathrm{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

~ 0: $\dfrac{\partial u}{\partial t} - \mathrm{div}_x(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$

$k = 4$:

~ -4: $\dfrac{\partial u}{\partial \tau} = 0.$

~ -3: $\dfrac{\partial u_1}{\partial \tau} = 0.$

~ -2: $\dfrac{\partial u_2}{\partial \tau} - \mathrm{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

~ -1: $-\mathrm{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

~ 0: $\dfrac{\partial u}{\partial t} - \mathrm{div}_x(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$

$k > 4$:

$\sim -k$: $\dfrac{\partial u}{\partial \tau} = 0.$

$\sim -k+1$: $\dfrac{\partial u_1}{\partial \tau} = 0.$

$\sim -k+2$: $\dfrac{\partial u_2}{\partial \tau} = 0.$

~ -2: $-\mathrm{div}_{y_2}(\tilde{a}(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

~ -1: $-\ \mathrm{div}_{y_1}(\tilde{a}(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0.$

~ 0: $\dfrac{\partial u}{\partial t} - \mathrm{div}_x(\tilde{a}(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$

where

$$\tilde{a}(x,t) = \int_0^1 a(x, y_1, y_2, t, \tau)\, d\tau.$$

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