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PREPRINT 2003–25

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CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg Sweden 2003

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Göteborg, December 2003

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NO 2003–25
ISSN 1404–4382

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Printed in Sweden
Chalmers University of Technology
Göteborg, Sweden 2003

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Abstract

A general concept of two-scale convergence is introduced and two-scale compactness theorems are stated and proved for some classes of non-periodic bounded functions in $L^2(\Omega)$. Further the relation to the classical notion of compensated compactness and the recent concept of two-scale compensated compactness is discussed and a defect measure for two-scale convergence is introduced.

1 Introduction

In 1989 Nguetseng [15] presented a new approach for the homogenization of partial differential equations, the so-called *two-scale convergence method*. The name two-scale convergence was introduced in [1]. Nguetseng's method has been widely used and has been developed in various ways. It has been applied to a variety of problems, see e.g. the recent survey [14] by Lukkassen et al. Let us in particular mention [4] where Amar proves two-scale compactness for a sequence of functions defined on BV. The extension to the almost periodic case is found in Casado-Diaz and Gayto [7], and in [6] Bourgeat et al develop a stochastic two-scale convergence (in the mean). Further, in Allaire and Briane [3] two-scale convergence is extended to the linear stationary multiscale case, and in Lions et al [13] to

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the monotone stationary multiscale case. In [11] the multiscale homogenization theorem is proved for parabolic problems. We also refer to [12] for a recent unified approach to homogenization in perforated domains. The two-scale convergence method relies on the sequential matching between a bounded sequence $\{u_h\}$ of functions in $L^2(\Omega)$ and a sequence $\{\nu_h\}$ of functions defined through $\nu_h(x) = v(x, \frac{x}{\varepsilon_h})$, $v \in L^2(\Omega \times Y)$. The original result by Nguetseng says that for v sufficiently smooth and Y-periodic in the second argument it holds upp to a subsequence that

$$\lim_{\varepsilon \to 0} \int_\Omega u_h(x) v(x, \frac{x}{\varepsilon_h})\, dx = \int_\Omega \int_Y u_0(x,y) v(x,y)\, dxdy.$$

The purpose of the present paper is to state and prove general compactness results that do not depend upon any periodicity assumptions on the test functions. We discuss it in the context of the classical compensated compactness by Murat and Tartar and the very recent concept of two-scale compensated compactness by Svanstedt and Wellander. We will see how the additional scale, which appears in the limit, allows us to relax the regularity needed to achieve results of compensated compactness type.

2 Weak convergence and general two-scale convergence

Let us first consider the elementary case of weak compactness in $L^2(\Omega)$. For $\{u_h\}$ a bounded sequence in $L^2(\Omega)$ it is well known that, up to a subsequence, $\{u_h\}$ converges weakly, i.e. for some $u \in L^2(\Omega)$ it holds that

$$\int_\Omega u_h(x)\nu(x)dx \to \int_\Omega u(x)\nu(x)dx$$

for all $v \in L^2(\Omega)$. Replacing ν with a bounded sequence $\{\nu_h\}$ in $L^2(\Omega)$ the situation gets less obvious. Depending on how the involved sequences of functions are chosen it may or may not hold that

$$\int_\Omega u_h(x)\nu_h(x)dx \to \int_\Omega u(x)\nu(x)dx, \tag{1}$$

where u and ν are the weak limits for $\{u_h\}$ and $\{\nu_h\}$. The easiest way to make sure that (1) holds is of course to make the further assumption that $\{\nu_h\}$ converges strongly. However, the notion of two-scale convergence provides us with an alternative. For suitable choices of v we have already noticed that

$$\int_\Omega u_h(x)\nu_h(x)dx \to \int_\Omega \int_A u_0(x,y)v(x,y)dxdy,$$

where $\nu_h(x) = v(x, \frac{x}{\varepsilon_h})$ is bounded in $L^2(\Omega)$ but not necessarily strongly convergent to any limit in $L^2(\Omega)$. It seems like the extra scale supports the convergence in cases where neither $\{u_h\}$ nor $\{\nu_h\}$ is strongly convergent.

A very natural question to ask is whether there are other ways to generate weakly convergent sequences $\{\nu_h\}$ such that (1) is true. To find out we investigate sequences of integral expressions of the type

$$\int_\Omega u_h(x)\nu_h(x)dx = \int_\Omega u_h(x)\tau_h v(x)dx,$$

where

$$\tau_h : X \to L^2(\Omega)$$

and $X \subset L^2(\Omega \times A)$, where $\Omega \subset \mathbb{R}^N$ and $A \subset \mathbb{R}^M$ are open and bounded. We show that results of the same kind as (1) holds under general assumptions on the operator τ_h and the admissible set X. One of the advantages with two-scale convergence is that all the properties one seeks for in the sequence $\{u_h\}$ are lifted out by the suitable choice of test functions. Therefore the characterization of the admissibility of test functions is one of the key problems in two-scale convergence. We will prove two theorems, Theorem 1 and Theorem 6, where the admissible of test functions belongs to two different subspaces of $L^2(\Omega \times A)$. One is based on separability and the other one characterized by its geometrical cone properties (Hahn-Banach).

Theorem 1 *Let Ω and A be bounded open subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively and assume that $\{u_h\}$ is a uniformly bounded sequence in $L^2(\Omega)$. Then*

$$\int_\Omega u_h(x)\nu_h(x)dx \to \int_\Omega \int_A u_0(x,y)v(x,y)dxdy,$$

if $\nu_h = \tau_h v$, $v \in X$, where $X \subset L^2(\Omega \times A)$ is a separable Banach space and

$$\tau_h : X \to L^2(\Omega)$$

has the properties

$$\lim_{h\to\infty} \|\tau_h v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega\times A)} \tag{2}$$

and

$$\|\tau_h v\|_{L^2(\Omega)} \leq C \|v\|_X . \tag{3}$$

We are now ready to define two-scale convergence in its generalized setting.

Definition 2 *A sequence $\{u_h\}$ in $L^2(\Omega)$ is said to two-scale converge to $u_0 \in L^2(\Omega \times A)$ with respect to $\{\tau_h\}$ if $\tau_h : X \to L^2(\Omega)$ and*

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)dx = \int_\Omega \int_A u_0(x,y)v(x,y)dxdy$$

for all $v \in X$.

Proof. We introduce

$$(F_h, v)_{X',X} = \int_\Omega u_h(x)\tau_h v(x)dx.$$

Clearly, according to the Hölder inequality,

$$(F_h, v)_{X',X} \leq C \left\| \tau_h v \right\|_{L^2(\Omega)} \tag{4}$$

and hence (3) yields that

$$(F_h, v)_{X',X} \leq C \left\| \tau_h v \right\|_{L^2(\Omega)} \leq D \left\| v \right\|_X . \tag{5}$$

(5) means that there exists a weakly$*$ convergent subsequence of $\{F_h\}$ in $X^{'}$ such that

$$(F_h, v)_{X',X} \to (F, v)_{X',X}$$

for all $v \in X$. Further, (2), (4), and a passage to the limit yields

$$(F, v)_{X',X} \leq C \left\| v \right\|_{L^2(\Omega \times A)}$$

and thus, by using the Hahn-Banach theorem, there exists a bounded linear functional $G \in (L^2(\Omega \times A))^{'}$ such that

$$(F, v)_{X',X} = (G, v)_{(L^2(\Omega\times A))',L^2(\Omega\times A)}$$

for all $v \in X$. Finally, according to L^2-duality, there exists $u_0 \in L^2(\Omega \times A)$ such that

$$(G, v)_{(L^2(\Omega\times A))',L^2(\Omega\times A)} = \int_\Omega \int_A u_0(x,y)v(x,y)dxdy$$

and the proof is complete. ■

Below we prove that the second scale in the two-scale limit is lost if we assume that $\{u_h\}$ converges strongly in $L^2(\Omega)$.

Proposition 3 *Let $\{\tau_h\}$ be like in Theorem 1 with the additional condition*

$$\nu_h = \tau_h v \rightharpoonup \nu = \int_A v(x,y)dyinL^2(\Omega) \tag{6}$$

and assume that $\{u_h\}$ converges strongly to u in $L^2(\Omega)$. Then

$$\int_\Omega u_h(x)\tau_h v(x)\,dx \to \int_\Omega \int_Y u(x)v(x,y)\,dxdy = \int_\Omega u(x)\nu(x)\,dx$$

for all admissible v.

Proof. We note that

$$\left|\int_\Omega u_h(x)\tau_h v(x)dx - \int_\Omega u(x)\tau_h v(x)\,dx\right| \leq \|u_h - u\|_{L^2(\Omega)}\,\|\tau_h v\|_{L^2(\Omega)} \to 0.$$

By condition (6) it holds that

$$\int_\Omega u(x)\tau_h v(x)\,dx \to \int_\Omega\int_A u(x)v(x,y)\,dxdy = \int_\Omega u(x)\nu(x)\,dx$$

and the proof is complete. ■

Remark 4 *The result in the proposition above demonstrates explicitly how the additional scale in the limits allows for convergence under conditions that would be unsufficient otherwise. If $\{u_h\}$ should be only weakly convergent we could not any more conclude that*

$$\int_\Omega u_h(x)\,\nu_h(x)dx \to \int_\Omega u(x)\nu(x)\,dx$$

while the two-scale equivalent

$$\int_\Omega u_h(x)\nu_h(x)\,dx \to \int_\Omega\int_Y u(x,y)v(x,y)\,dxdy$$

would still hold.

For the proof of the second result (Theorem 6), we first need to recall the following version of the Hahn-Banach theorem.

Lemma 5 (Hahn-Banach) *Let X be a normed linear space and Y a subset to X. Further, assume that $f : Y \to \mathbb{R}$ is linear and that*

$$\left|\sum_{i=1}^{n} c_i f(v_i)\right| \leq C\left\|\sum_{i=1}^{n} c_i v_i\right\|_X$$

for some C and all $v_i \in Y, c_i \in \mathbb{R}$.

Then there exists a linear functional g that extends f from Y to X and with $\|g\|_{X'} \leq C$.

Proof. Put $p(v) = C\,\|v\|_X$ in Theorem 2.3.1 in Edwards [9] ■

We are now ready to state and prove

Theorem 6 *Let Ω and A be bounded open subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, and assume that $\{u_h\}$ is a uniformly bounded sequence in $L^2(\Omega)$, X a subset contained in $L^2(\Omega \times A)$ endowed with the norm of $L^2(\Omega \times A)$, and*

$$\tau_h : X \to L^2(\Omega)$$

a sequence of linear maps, such that, for some C independent of h,

$$\left\| \sum_{i=1}^{n} c_i \tau_h v_i \right\|_{L^2(\Omega)} \leq C \left\| \sum_{i=1}^{n} c_i v_i \right\|_{L^2(\Omega \times A)} \tag{7}$$

for all $v_i \in X, c_i \in \mathbb{R}$. Then, for some $u_0 \in L^2(\Omega \times A)$ and up to a subsequence,

$$\lim_{h \to \infty} \int_\Omega u_h(x) \tau_h v(x)\, dx = \int_\Omega \int_A u_0(x,y) v(x,y)\, dxdy$$

holds for all $v \in X$.

Proof. We introduce

$$(F_h, v)_{X',X} = \int_\Omega u_h(x) \tau_h v(x) dx.$$

Clearly, by (7) and the Hölder inequality,

$$\left| \sum_{i=1}^{n} c_i (F_h, v_i)_{X',X} \right| = \left| \sum_{i=1}^{n} c_i \int_\Omega u_h(x)(\tau_h v_i)(x) dx \right| \leq \tag{8}$$

$$\leq C \left\| \sum_{i=1}^{n} c_i \tau_h v_i \right\|_{L^2(\Omega)} \leq D \left\| \sum_{i=1}^{n} c_i v_i \right\|_{L^2(\Omega \times A)}.$$

(8) and Lemma 5 yield that there exists an extension G^h of F^h such that

$$(G_h, v)_{(L^2(\Omega \times A))', L^2(\Omega \times A)} \leq D \left\| v \right\|_{L^2(\Omega \times A)}. \tag{9}$$

(9) and the separability of $L^2(\Omega \times A)$ imply that there exists a weakly$*$ convergent subsequence of $\{G^h\}$ in $(L^2(\Omega \times A))'$ such that

$$(G_h, v)_{(L^2(\Omega \times A))', L^2(\Omega \times A)} \to (G, v)_{(L^2(\Omega \times A))', L^2(\Omega \times A)}$$

for all $v \in L^2(\Omega \times A)$. Finally, according to L^2-duality, there exists $u_0 \in L^2(\Omega \times A)$ such that

$$(G, v)_{(L^2(\Omega \times A))', L^2(\Omega \times A)} = \int_\Omega \int_A u_0(x,y) v(x,y) dxdy$$

and therefore, for the restriction F of G to X,

$$(F, v)_{X',X} = \int_\Omega \int_A u_0(x,y) v(x,y) dxdy$$

for any $v \in X$. ∎

Remark 7 *All the results in this chapter are easily generalized from L^2 to the L^p-case when $p > 1$. The case $p = 1$ has to be handled in a somewhat different manner. Since $L^1(\Omega)$ is not reflexive, we can not apply weak sequential compactness. We can however argue as this. Let $C_c^0(\Omega)$ denote the set of continuous functions with compact support in Ω. Then it is well known that its dual $(C_c^0(\Omega))' = M(\Omega)$ i.e. the space of Radon measures on Ω. Let us now as usual identify $L^1(\Omega)$ with a subspace of $M(\Omega)$. It follows that, if $\{u_h\}$ is a sequence which is uniformly bounded in $L^1(\Omega)$ and if*

$$\tau_h : C_c^0(\Omega \times A) \to C_c^0(\Omega)$$

is a sequence of maps, such that,

$$\lim_{h\to\infty} \|\tau_h v\|_{C_c^0(\Omega)} \leq C \|v\|_{C_c^0(\Omega\times A)}. \tag{10}$$

Then there exists a Radon measure $\mu_0 \in M(\Omega \times A)$ and a subsequence, still denoted $\{u_h\}$, in $L^1(\Omega)$ such that,

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)\,dx = \langle \mu_0(x,y), v(x,y)\rangle_{M(\Omega\times A),C_c^0(\Omega\times A)}.$$

If we can argue that the limit element μ_0 actually belongs to $L^1(\Omega \times A)$, then the two-scale convergence is compact in L^1. Weak two-scale compactness in L^1 can also be proved using the usual Dunford-Pettis characterization. For a complete exposition of periodic two-scale convergence of Radon measures we refer to [4].

Remark 8 *An important question for applications is when a bounded sequence in L^2 is actually an admissible set. I.e. when can we expect the following convergence to hold:*

$$\lim_{h\to\infty} \left\| v(x, \frac{x}{\epsilon_h}) \right\|_{L^2(\Omega;\mathbb{R}^3)} = \|v\|_{L^2(\Omega\times Y;\mathbb{R}^3)}, \tag{11}$$

In [5] this question is answered: Namely, if $\{v(\cdot, \frac{\cdot}{\epsilon_h})\}$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^3)$, if $\{\epsilon_h \operatorname{div} v(\cdot, \frac{\cdot}{\epsilon_h})\}$ and $\{\epsilon_h \operatorname{curl} v(\cdot, \frac{\cdot}{\epsilon_h})\}$ are uniformly bounded in $L^2(\Omega;\mathbb{R})$ and $L^2(\Omega;\mathbb{R}^3)$ respectively, then $\{v(\cdot, \frac{\cdot}{\epsilon_h})\}$ is an admissible set in the sense of (11).

Remark 9 (Periodic case) *Let $\tau_h v(x) = v(x, x/\epsilon_h)$ where $v(x,y)$ is periodic (unit period for instance) and Lebesgue measurable in the second argument, and where ϵ_h is a sequence of positive numbers tending to zero as h tends to $+\infty$. Then (11) becomes*

$$\lim_{h\to\infty} \left\| v(x, \frac{x}{\epsilon_h}) \right\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega\times Y)}, \tag{12}$$

and, for some $u_0 \in L^2(\Omega \times Y)$ and up to a subsequence,

$$\lim_{h\to\infty} \int_\Omega u_h(x) v(x, \frac{x}{\epsilon_h})\,dx = \int_\Omega \int_Y u_0(x,y)v(x,y)\,dxdy,$$

where $Y =]0,1[^M$. Typical examples of admissible test functions as above are those in $L^2(\Omega; C_{per}(Y))$ and, for Ω bounded, $L^2_{per}(Y; C(\overline{\Omega}))$. In fact, for these function spaces, (12) holds with equality and with $C = 1$.

Remark 10 (Periodic multiscale case) *Let* $\tau_h v(x) = v(x, x/\epsilon_h^1, \dots, x/\epsilon_h^q)$ *where* $v(x, y_1, \dots, y_q)$ *is periodic (unit period for instance) and Lebesgue measurable in* $y_1, \dots, y_q$*, and where* ϵ_h *is a sequence of positive numbers tending to zero as* h *tends to* $+\infty$*. Then (11) becomes*

$$\lim_{h\to\infty} \left\| v(x, \frac{x}{\epsilon_h^1}, \dots, \frac{x}{\epsilon_h^q}) \right\|_{L^q(\Omega)} \le C \left\| v \right\|_{L^q(\Omega \times Y_1 \times \ldots \times Y_q)},$$

and, for some $u_0 \in L^p(\Omega \times Y_1 \times \ldots \times Y_q)$ *and up to a subsequence,*

$$\lim_{h\to\infty} \int_\Omega u_h(x) v(x, \frac{x}{\epsilon_h^1}, \dots, \frac{x}{\epsilon_h^q})\, dx =$$

$$\int_\Omega \int_{Y_1} \dots \int_{Y_q} u_0(x, y_1, \dots, y_q)) v(x, y_1, \dots, y_q))\, dx dy_1 \dots dy_q,$$

where $Y_i =]0,1[^M,\ i = 1, \dots, q$

3 Two-scale convergence and compensated compactness

In the Theorems 1 and 6 we did not ask anything more than boundedness in $L^2(\Omega)$ from $\{u_h\}$, while we made more specific assumptions on how the sequence $\{v_h\}$ should have appeared. A quite famous result by Murat and Tartar, the *div-curl* lemma, adresses a similar situation under somewhat different assumptions. Here the assumptions are strengthened in addition to boundedness in $L^2(\Omega)$ on both sequences by imposing constraints on certain arrangements of the partial derivatives of $\{u_h\}$ and $\{v_h\}$. In [16] Tartar utilizes this study and proves general compactness results for quadratic forms $Q(u_h)$ under the name of *compensated compactness*. The div-curl lemma reads:

Theorem 11 *Let* $\{u_h\}$ *and* $\{v_h\}$ *be bounded sequences in* $[L^2(\Omega)]^N$ *and* u *and* v *the weak limits of suitable subsequences. If, in addition,* $\{\operatorname{div} u_h\}$ *and* $\{\operatorname{curl} v_h\}$ *are, possibly up to a further subsequence, strongly convergent in* $W^{-1,2}(\Omega)$*, then*

$$\lim_{h\to\infty} \int_\Omega u_h(x) \cdot \nu_h(x) \varphi(x) dx = \int_\Omega u(x) \cdot v(x) \varphi(x) dx$$

for any $\varphi \in D(\Omega)$*.*

Recently Tartar's compensated compactness result has been extended to the method of two-scale convergence in [5] by Wellander, Birnir and Svanstedt. The two-scale version of the div-curl lemma reads:

Theorem 12 *Let* $\{u_h\}$ *and* $\{\nu_h\}$ *be bounded sequences in* $[L^2(\Omega)]^N$ *and denote by* u_0 *and* v_0 *the weak two-scale limits of suitable subsequences. If, in addition,* $\varepsilon_h\{\operatorname{div} u_h\}$ *and* $\varepsilon_h\{\operatorname{curl} \nu_h\}$ *are bounded in* $L^2(\Omega)$*, then, possibly up to a further subsequence,*

$$\lim_{h\to\infty}\int_\Omega u_h(x)\cdot\nu_h(x)\varphi(x,\frac{x}{\varepsilon_h})dx=\int_\Omega\int_Y u_0(x,y)\cdot v_0(x,y)\varphi(x,y)dxdy$$

for any $\varphi\in D(\Omega;C_\sharp^\infty(Y))$.

Remark 13 *Note that we may also consider this like that the second scale allows for the convergence of the product of three weakly convergent sequences in the sense that*

$$\int_\Omega u_h(x)\cdot\nu\lim_{h\to\infty}{}_h(x)\varphi_h(x)dx=\int_\Omega\int_Y u_0(x,y)\cdot v(x,y)\varphi(x,y)dxdy,$$

where $\varphi_h(x)=\varphi(x,\frac{x}{\varepsilon_h})$

Remark 14 *The relationship betweens Theorems 11 and 12 deserves some attention. It is easy to see that Theorem 12 leads to the result in Theorem 11 if we chose* $\varphi(x,y)=\varphi(x)$ *and*

$$\int_Y u_0(x,y)\cdot v(x,y)dy=\int_Y u_0(x,y)dy\cdot\int_Y v(x,y)dy. \tag{13}$$

If for example $u_0(x,y)=u(x)$ *the identity (13) follows immediately from the fact that the weak limit* u *is obtained from the two-scale limit* u_0 *through*

$$u(x)=\int_Y u_0(x,y)dy.$$

It is well known that the loss of the second scale appears when $\{u_h\}$ *is strongly convergent. However, for this case the result in Theorem 11 appears trivially by elementary functional analysis. An important question in this connection is whether the second scale may vanish under some conditions not including strong convergence. We demonstrate a such situation below. By the assumption that* $\{\operatorname{curl} v_h\}$ *is bounded in* $L^2(\Omega)$ *we conclude from two-scale compactness that* $\operatorname{curl}_y v_0=0$*. Classical vector calculus arguments then says that there exists a function* Ψ *such that*

$$v_0(x,y)=v(x)+\nabla_y\Psi(x,y).$$

Moreover $\operatorname{div}_y u_0=0$ *and this together with integration by parts applied on the second term yields*

$$\int_\Omega(\int_Y u_0(x,y)\cdot(v(x)+\nabla_y\Psi(x,y))dy)\varphi(x)dx=\int_\Omega(\int_Y u_0(x,y)dy)\cdot v(x)\varphi(x)dx.$$

Thus, by letting

$$u(x)=\int_Y u_0(x,y)dy.$$

this gives the identity

$$\int_\Omega(\int_Y u_0(x,y)\cdot v_0(x,y)dy)\varphi(x)dx=\int_\Omega u(x)\cdot v(x)\varphi(x)dx$$

for all $\varphi\in D(\Omega)$.

4 Generalized two-scale convergence and defect measures

In this final section we exhibit an explicit example of a sequence $\{\tau_h\}$ of operators of the type introduced in Theorem 1 and investigate the relationship with well known cases. Outgoing from this we introduce an approach that can be seen as a type of generalized unfolding (see [8]) and make some preliminary observations concerning how to introduce a defect measure for generalized two-scale convergence.

4.1 A special case for general two-scale convergence

Let us consider again the expression

$$\int_\Omega u_h(x)\,\nu_h(x)\,dx.$$

If we let $\{\nu_h\}$ appear through a sequence of Hilbert-Schmidt operators $\{\tau_h\}$ like

$$\nu_h(x) = \tau_h v(x) = \int_Y w_h(y)v(x,y)dy,$$

where $\{w_h\}$ is bounded in $L^2(\Omega)$, then $\{\nu_h\}$ converges strongly in $L^2(\Omega)$ (see([2]) 8.9) and hence

$$\int_\Omega u_h(x)\,\nu_h(x)\,dx \to \int_\Omega u(x)\,v(x)\,dx.$$

For the corresponding operator with two scales given by

$$\nu_h(x) = \tau_h v(x) = \int_Y w_h(x,y)v(x,y)dy,$$

where $\{w_h\}$ is bounded in $L^2(\Omega \times A)$, the strong convergence for $\{\nu_h\}$ does not hold in general. If we chose e.g.

$$w_h(x,y) = w_h^1(x)w^2(y)$$

with $\{w_h^1\}$ bounded in $L^2(\Omega)$ and $w^2 \in L^2(A)$, the convergence of $\{\nu_h\}$ in $L^2(\Omega)$ will be no stronger than that of $\{w_h^1\}$. However, still it is strong enough to allow for generalized two-scale convergence in the sense of (14) below.

We are now ready to show that, under assumptions neither including strong convergence in any Lebesguespace nor differentiability or continuity requirements on the involved functions, it holds up to a subsequence that

$$\lim_{h\to\infty}\int_\Omega u_h(x)\,\nu_h(x)\,dx = \int_\Omega\int_A u_0(x,y)\,v(x,y)\,dxdy \tag{14}$$

where $u_0 \in L^2(\Omega \times A)$ and

$$u_h \rightharpoonup u = \int_A u_0(x,y)\,dy \text{ in } L^2(\Omega). \tag{15}$$

This is true if $\{u_h\}$ and $\{\nu_h\}$ are bounded in $L^2(\Omega)$ and

$$\nu_h(x) = \tau_h v(x) = \int_A w_h(x,y)\,v(x,y)\,dxdy$$

for some $v \in X = L^4(\Omega \times A)$, and $\{w_h\}$ a bounded sequence in X such that up to a subsequence

$$w_h \rightharpoonup w \text{ in } L^4(\Omega \times A)$$

and

$$w_h^2 \rightharpoonup W \text{ in } L^2(\Omega \times A),$$

where $W \in L^\infty(\Omega \times A)$.

$$\int_A w_h(x,y)\,dy = 1.$$

The reason for this last condition is that the average over A for the two-scale limit u_0 shall coincide with the corresponding weak limit u. It remains to show that

$$\lim_{h\to\infty} \|\tau_h v\|_{L^2(\Omega)} \leq C\,\|v\|_{L^2(\Omega\times A)} \tag{16}$$

and

$$\|\tau_h v\|_{L^2(\Omega)} \leq C\,\|v\|_X\,. \tag{17}$$

The weak convergence of w_h^2 in $L^2(\Omega \times A)$ together with Jensens inequality yields that (16) now becomes

$$\begin{aligned}
\|\tau_h v\|^2_{L^2(\Omega)} &= \left\| \int_A w_h(x,y)\,v(x,y)\,dy \right\|^2_{L^2(\Omega)} = \int_\Omega (\int_A w_h(x,y)\,v(x,y)\,dy)^2 dx \\
&\leq \int_\Omega \int_A w_h^2(x,y)\,v^2(x,y)\,dxdy \to \int_\Omega \int_A W(x,y)\,v^2(x,y)\,dxdy \\
&\leq C\,\|v\|^2_{L^2(\Omega\times A)}
\end{aligned}$$

Clearly

$$\begin{aligned}
\|\tau_h v\|^2_{L^2(\Omega)} &= \int_\Omega \int_A w_h^2(x,y) v^2(x,y)\,dydx \\
&\leq \|w_h\|^2_{L^4(\Omega\times A)} \|v\|^2_{L^4(\Omega\times A)} \leq C\,\|v\|^2_{L^4(\Omega\times A)}
\end{aligned}$$

and hence also (17) is proven. For any $v = v(x) \in L^2(\Omega)$ we obtain

$$\lim_{h\to\infty} \int\limits_{\Omega} u_h(x)\,\nu_h(x)\,dx = \lim_{h\to\infty} \int\limits_{\Omega} u_h(x) \int\limits_{A} w_h(x,y)\,v(x)\,dydx$$

$$= \lim_{h\to\infty} \int\limits_{\Omega} u_h(x)\,\nu(x)\,dx = \int\limits_{\Omega} \int\limits_{A} u_0(x,y)\,v(x)\,dxdy = \int\limits_{\Omega} (\int\limits_{A} u_0(x,y)\,dy) v(x)\,dx$$

and thus (15) is proven. Moreover, if we assume that $w \equiv 1$ it easy to show that

$$\nu_h \rightharpoonup \int_A v(x,y)dy \ \text{ weakly in } \ L^2(\Omega).$$

Remark 15 *Changing the order of integration it is possible to reformulate the left hand side of (14) into weak convergence in $L^{\frac{4}{3}}(\Omega \times A)$ like*

$$\langle u_h, \tau_h v\rangle_{L^2(\Omega)} = \int\limits_{\Omega} u_h(x)\,\tau_h v(x)\,dx = \int\limits_{\Omega} \int\limits_{A} u_h(x)\,w_h(x,y)\,v(x,y)\,dydx,$$

where $v \in L^4(\Omega \times A)$ and $\{u_h w_h\}$ is bounded in $L^{\frac{4}{3}}(\Omega \times A)$. We can look upon this as if we had "unfolded" u_h by means of the adjoint operator

$$\tau_h^* u_h(x,y) = u_h(x)\,w_h(x,y)\,.$$

It is now possible to perform the two-scale convergence process in the equivalent form

$$\langle \tau_h^* u_h, v\rangle_{L^2(\Omega\times Y)} = \int\limits_{\Omega} \int\limits_{A} \tau_h^* u_h(x,y)\,v(x,y)\,dxdy$$

$$= \int\limits_{\Omega} \int\limits_{A} u_h(x)\,w_h(x,y)\,v(x,y)\,dxdy \to \int\limits_{\Omega} \int\limits_{A} u_0(x,y)\,v(x,y)\,dxdy.$$

Note that the limit $u_0 \in L^2(\Omega \times A)$ eventhough it is the weak limit of a sequence that is just bounded in $L^{\frac{4}{3}}(\Omega \times A)$.

The remark below provides us with a link between the type of two-scale convergence introduced in this section and traditional periodic two-scale convergence.

Remark 16 *Let $v \in C^\infty(\Omega; C^\infty_{per}(Y))$, where $Y = (0,1)^n$, i.e. the set of smooth functions with (unit) period in the second argument. We can construct the usual set of periodic oscillating test functions $v(x, x/\varepsilon_h)$ by setting*

$$v_h(x) = \tau_h v(x) = \int_Y \delta_{x/\varepsilon_h}(y) v(x,y)dy = v(x, x/\varepsilon_h),$$

where δ_x is the usual delta distribution.

4.2 Defect measures for two-scale convergence

A general complication with two-scale convergence is that the sequence to be analyzed and the corresponding two-scale limit lives in completely different spaces. The operator τ_h^* helps us to overcome this problem. Among others it makes it possible to introduce a defect measure ς_D for two-scale convergence. Below we suggest a defect measure and show how it can be simplified when the two-scale limit u_0 belongs to the admissible space X. We compare $\tau_h^* u_h$ with the two-scale limit in the norm topology of $L^2(\Omega \times A)$.

$$
\begin{aligned}
\varsigma_D(u_h, u_0) &= \lim_{h\to\infty} \|\tau_h^* u_h - u_0\|^2_{L^2(\Omega\times A)} \\
&= \lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2 - 2\tau_h^* u_h(x,y)\, u_0(x,y) + u_0^2(x,y)\, dxdy \\
&= \lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2 - 2u_h(x)\, \tau_h u_0(x) + u_0^2(x,y)\, dxdy \\
&= \lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2 - u_0^2(x,y)\, dxdy
\end{aligned}
$$

if $u_0 \in X$. In this case we have

$$
\lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2 - u_0^2(x,y)\, dxdy
$$

as a measure on what is missing to obtain strong convergence. The properties of such defect measures and under which conditions τ_h^* and its possible generalizations exists with appropiate characteristics will be scrutinized in a forthcoming paper.

We close this section by presenting an explicit example of unfolding operators in the periodic setting.

Example This example is due to Cioranescu et. al. [8]. Let $Y = (0,1)^n$ and let $\{\epsilon_h\}$ be a sequence of positive real numbers tending to zero as $h \to \infty$. For any $x \in \mathbb{R}^n$ we write

$$
x = \epsilon_h \left(\left[\frac{x}{\epsilon_h} \right]_Y + \left\{ \frac{x}{\epsilon_h} \right\}_Y \right),
$$

where $[\cdot]$ denotes the integer part and where we use the fact that for any $x \in \mathbb{R}^n$ and $h \in \mathbb{N}$ there exists a unique number $k \in \mathbb{Z}^n$ such that

$$
\frac{x}{\epsilon_h} = k + y, \quad y \in Y.
$$

For any $u \in L^2(\Omega)$ we now define the unfolding operators in the periodic setting. See also

$$
\tau_h^* : L^2(\Omega) \to L^2(\Omega \times Y)
$$

as

$$\tau_h^* u(x,y) = u\left(\epsilon_h \left[\frac{x}{\epsilon_h}\right]_Y + \epsilon_h y\right).$$

Acknowledgement: The authors wish to thank prof. Francois Murat for inspiring discussions particularily concerning the relationship between classical and two-scale compensated compactness.

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